Measuring the stance of monetary policy in conventional and unconventional environments

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Abstract

This article introduces an idea for summarizing of the stance of monetary policy with quantities derived from a class of yield curve models that respect the zero lower bound constraint for interest rates. The "Effective Monetary Stimulus" aggregates the current and estimated expected path of interest rates relative to the neutral interest rate from the yield curve model. Unlike shadow short rates, Effective Monetary Stimulus measures are consistent and comparable across conventional and unconventional monetary policy environments, and are less subject to variation with modeling choices, as I demonstrate with two and three factor models estimated with different data sets. Full empirical testing of the inter-relationships between Effective Monetary Stimulus measures and macroeconomic data remains a topic for future work.

JEL: E43, E52, G12

Keywords: unconventional monetary policy; zero lower bound; shadow short rate; term structure model

1 Introduction

This article proposes a new measure of the stance of monetary policy derived from shadow/zero lower bound (ZLB) yield curve models. The motivation for what I call the "Effective Monetary Stimulus" (EMS measure) is to improve on aspects that have been questioned regarding the use of shadow short rates (SSRs) as a summary metric for the stance of unconventional monetary policy. The EMS measure also offers improvements on an alternative metric, i.e. the horizon to non-zero policy rates.

As background, SSRs obtained from shadow/ZLB yield curve models have been proposed as a measure of the stance of monetary policy in Krippner (2011, 2012, 2013b,d) as cited by Bullard (2012, 2013), and Wu and Xia (2013) as cited by Hamilton (2013) and Higgins and Meyer (2013) The proposal has intuitive appeal because when the SSR is positive it equals the actual short rate, but the SSR is free to evolve to negative levels after the actual short rate becomes constrained by the ZLB. Figure 1 illustrates the concept for the U.S. Federal Funds Rate (FFR) and an estimated shadow short rate from

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December 2008, when the FFR was set to a 0-0.25% target range. In essence, the SSR evolves like the short rate that would prevail in the absence of physical currency,[1] and so it can be used as an indicator of further policy easing beyond the zero policy rate. Quantitatively, comparing the unconventional/ZLB and conventional/pre-ZLB periods for the U.S., Claus, Claus, and Krippner (2013) show that the SSR responds to monetary policy shocks similarly to the FFR, and Wu and Xia (2013) show that the effects of the SSR on macroeconomic variables are similar to the FFR.

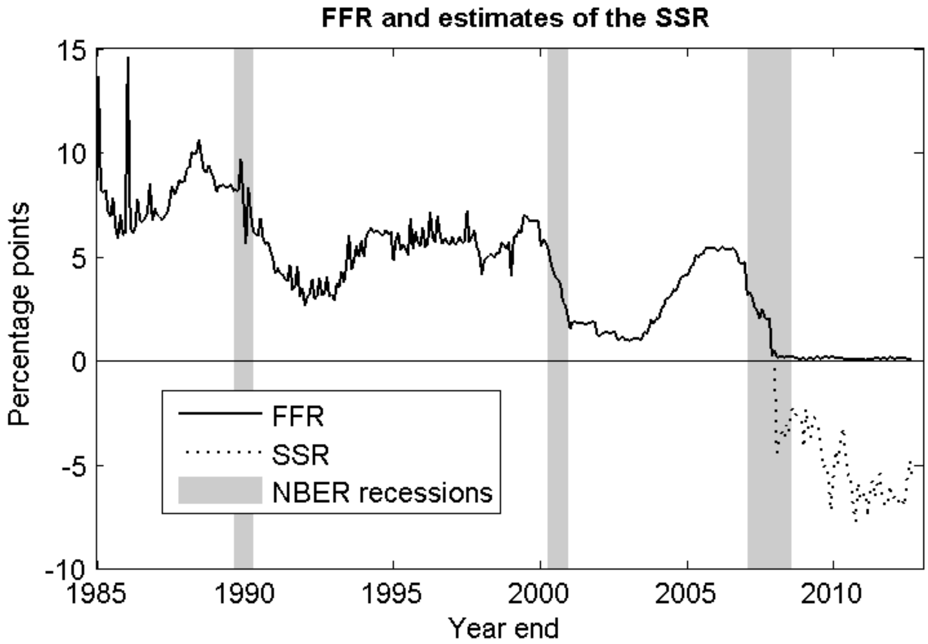

Figure 1: The Federal Funds Rate (FFR) and the estimated shadow short rate (SSR) from December 2008. The ZLB constrains the FFR essentially at zero, while the SSR can freely evolve to negative levels.

Nevertheless, negative SSRs are necessarily estimated quantities, and so they will vary with the practical choices underlying their estimation. In particular, it has been well-established in Christensen and Rudebusch (2013a,b), Bauer and Rudebusch (2013), and Krippner (2013d) that SSR estimates can be materially sensitive to the following choices: (1) the specification of the shadow/ZLB model (e.g. Black (1995) or Krippner (2011, 2012, 2013), two or three factors, parameter restrictions on the mean-reversion matrix, etc.); (2) the data used for estimation (e.g. using yield curve data out to maturities of 10 year or 30 years, and the sample period); and (3) the method used for estimation (e.g. the extended, iterated extended, or the unscented Kalman filters).[2]

In addition, from an economic perspective, negative shadow rates are not an actual interest rate faced by economic agents. That is, borrowers face current and expected interest rates that are based on the ZLB constraint (with appropriate margins), not negative interest rates (which would result in borrowers being paid the absolute interest rate by investors). As such, SSRs are not consistent and directly comparable across conventional/non-ZLB and unconventional/ZLB environments. To highlight this point,

[1]Physical currency effectively sets the ZLB for interest rates because it is always available as an alternative investment to bonds, but with a zero rate of return.

[2]Longer maturity spans and estimation with the iterated extended Kalman filter produce more negative SSR estimates; see Krippner (2013d).

figure 2 illustrates that easing the SSR from 5 to 0 percent provides more monetary policy stimulus than easing the SSR from 0 to -5 percent, because the entire yield curve moves down markedly in first case but the ZLB constrains declines in short- and mid-maturity interest rates in the second case.[3]

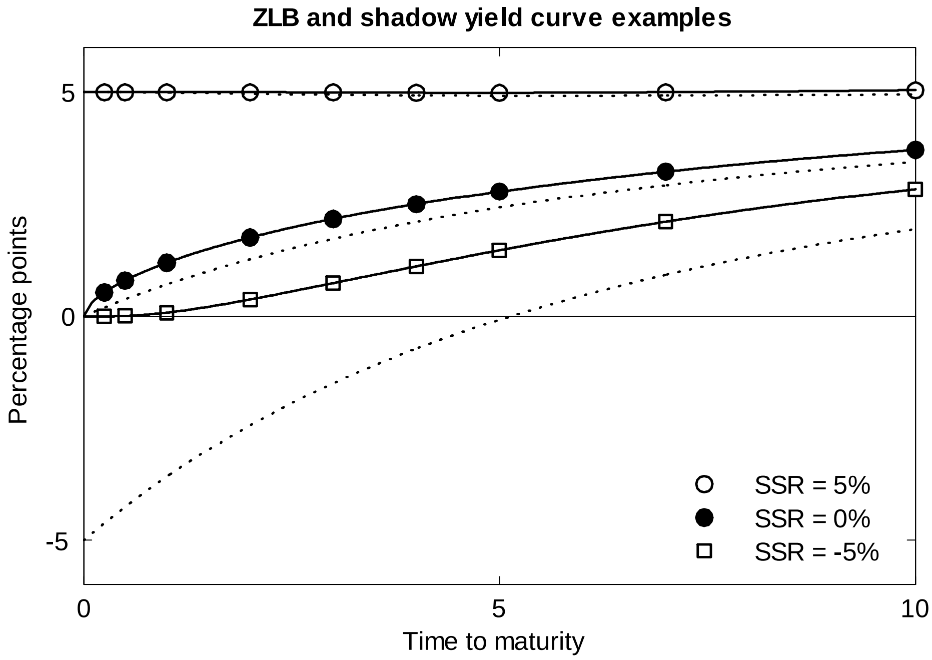

Figure 2: ZLB yield curves and shadow yield curves (dotted lines below ZLB yield curves) for different values of the SSR, while keeping the long-run yields constant. Monetary policy stimulus from the ZLB yield curve (i.e. declines in actual interest rates) is attenuated when the SSR adopts negative values.

As an alternative metric for unconventional monetary policy, Bauer and Rudebusch (2013) propose the "lift-off horizon", i.e. the median time for simulated future actual short rates from the estimated shadow/ZLB model to reach 0.25 percent. The lift-off horizon has proven more robust than the SSR to different model specifications, and it also provides a probabilistic summary measure of actual interest rates faced by economic agents for the given horizon, rather than the non-obtainable and therefore less economically relevant negative SSRs. However, the lift-off horizon does not provide an indication of what economic agents will face for longer horizons into the future, which should also influence their decisions, and neither does it have a conventional counterpart for comparison across non-ZLB and ZLB environments.

The EMS measure that I propose improves on the SSR and the lift-off horizon by directly summarizing current and expected actual interest rates relative to the neutral interest rate. Specifically, I obtain the expected path of the actual short rate and its long-run expectation from a shadow/ZLB Gaussian affine term structure model (hereafter GATSM) of the yield curve and then integrate the difference between those quantities over the horizon from zero to infinity. In ZLB periods, short rate expectations will initially include a period of zero followed by an non-zero path that converges to the long-run expectation, and in non-ZLB periods the expected path of the short rate is entirely non-zero as it converges to the long-run expectation. However, in both regimes, the EMS

[3] Figure 9 latter provides a more detailed perspective on how monetary policy stimulus attentuates as a function of the SSR.

measure aggregates expected short rates relative to their long-run expectation from the same estimated model, and so the EMS measure is directly comparable between ZLB and non-ZLB periods. The practical advantage of EMS measures is their robustness relative to SSRs; i.e. the EMS measures obtained from shadow/ZLB-GATSMs with two and three factors estimated with different data turn out to be very similar, while the corresponding SSR estimates are quite different.

The article proceeds as follows. Section 2 provides an overview of the Krippner (2011, 2012, 2013) and Black (1995) shadow/ZLB-GATSMs, respectively the K-GATSM and B-GATSM hereafter.[4] In section 3, I first outline how EMS measures may be obtained using the shadow-GATSM from either the K-GATSM or B-GATSM, and then apply the EMS framework to the K-GATSM results under the risk-adjusted measure from Krippner (2013d). Section 4 introduces and illustrates EMS measures under the physical measure. Section 5 discusses some ideas related to the EMS measure to follow up in future work, including potential improvements, empirical testing, and some conceptual questions. Section 6 concludes.

2 Overview of shadow/ZLB-GATSMs

In this section, I provide an overview of the K-GATSM and B-GATSM classes of models. The main objective from the perspective of the present article is to establish notation for the shadow-GATSM, which contains the component subsequently used to define the EMS measures in sections 3 and 5. I also briefly discuss in section 2.2 how the models may be estimated in principle so it is apparent how observed yield curve data and the specified model defines the EMS measures. Details and examples of actual estimations are available from the given references; in this article I simply use the results already available from Krippner (2013d) and some supplementary estimations to illustrate EMS measures in practice.

2.1 The shadow-GATSM term structure

I adopt the generic GATSM specification from Dai and Singleton (2002) pp. 437-38 to define the shadow-GATSM. Hence, the SSR is:

$$\mathrm{r}(t) = a_0 + b_0' x(t) \tag{1}$$

where a_0 is a constant, b_0 is a constant $N \times 1$ vector containing the weights for the N state variables $x_n(t)$, and $x(t)$ is an $N \times 1$ vector containing the N state variables $x_n(t)$. Under the physical $\mathbb{P}$ measure, $x(t)$ evolves as the following correlated vector Ornstein-Uhlenbeck process:

$$\mathrm{d}x(t) = \kappa [\theta - x(t)]\, \mathrm{d}t + \sigma \mathrm{d}W(t) \tag{2}$$

where θ is a constant $N \times 1$ vector representing the long-run level of $x(t)$, κ is a constant $N \times N$ matrix that governs the deterministic mean reversion of $x(t)$ to θ, σ is a constant $N \times N$ matrix representing the potentially correlated volatilities of $x(t)$, and $\mathrm{d}W(t)$ is an $N \times 1$ vector with independent Wiener components $\mathrm{d}W_n(t) \sim N(0,1)\sqrt{\mathrm{d}t}$. From Meucci

[4] The Wu and Xia (2013) model is a discrete-time version of the K-GATSM, although it is derived differently.

(2010) p. 3, the solution for the stochastic differential equation is:

$$x(t+\tau) = \theta + \exp(-\kappa\tau)[x(t) - \theta] + \int_t^{t+\tau} \exp(-\kappa[\tau - u])\,\sigma \mathrm{d}W(u) \quad (3)$$

which gives the following expectation, as at time t, under the $\mathbb{P}$ measure (also see Dai and Singleton (2002) p. 438):

$$\mathbb{E}_t[x(t+\tau)] = \theta + \exp(-\kappa\tau)[x(t) - \theta] \quad (4)$$

Therefore, what I will call the expected path of the SSR under the $\mathbb{P}$ measure, $\mathbb{E}_t[\mathrm{r}(t+\tau)]$, is:

$$\begin{aligned} \mathbb{E}_t[\mathrm{r}(t+\tau)] &= a_0 + b_0'\mathbb{E}_t[x(t+\tau)] \\ &= a_0 + b_0'\{\theta + \exp(-\kappa\tau)[x(t) - \theta]\} \end{aligned} \quad (5)$$

Note that the current SSR, $\mathrm{r}(t)$, is contained in $\mathbb{E}_t[\mathrm{r}(t+\tau)]$, i.e.:

$$\mathbb{E}_t[\mathrm{r}(t+\tau)]|_{\tau=0} = a_0 + b_0'x(t) = \mathrm{r}(t) \quad (6)$$

and so the current and expected SSRs do not need to be referred to separately (which also holds for $\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]$ below).

The market prices of risk are linear with respect to the state variables, i.e.:[5]

$$\Pi(t) = \sigma^{-1}[\gamma + \Gamma x(t)] \quad (7)$$

where γ and Γ are respectively a constant $N \times 1$ vector and constant $N \times N$ matrix. The risk-adjusted process for $x(t)$ is:

$$\mathrm{d}x(t) = \tilde{\kappa}\left[\tilde{\theta} - x(t)\right]\mathrm{d}t + \sigma \mathrm{d}\tilde{W}(t) \quad (8)$$

where $\tilde{\kappa} = \kappa + \Gamma$ and $\tilde{\theta} = \tilde{\kappa}^{-1}(\kappa\theta - \gamma)$.

Shadow forward rates for the shadow-GATSM are:

$$\mathrm{f}(t,\tau) = \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] + \mathrm{VE}(\tau) \quad (9)$$

where $\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]$ is the expected path of the SSR under the risk-adjusted $\mathbb{Q}$ measure:

$$\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] = a_0 + b_0'\left\{\tilde{\theta} + \exp(-\tilde{\kappa}\tau)\left[x(t) - \tilde{\theta}\right]\right\} \quad (10)$$

and $\mathrm{VE}(\tau)$ is the forward rate volatility effect from Krippner (2013d) appendix I:

$$\mathrm{VE}(\tau) = \frac{1}{2}\left[\int_0^\tau b_0' \exp(-\tilde{\kappa}\tau)\,\mathrm{d}s\right]\sigma\sigma'\left[\int_0^\tau \exp(-\tilde{\kappa}'\tau)\,b_0\,\mathrm{d}s\right] \quad (11)$$

The expression for shadow interest rates $\mathrm{R}(t,\tau)$ is defined using the standard continuous-time term structure relationships,[6] i.e.:

$$\mathrm{R}(t,\tau) = \frac{1}{\tau}\int_0^\tau \mathrm{f}(t,u)\,\mathrm{d}u \quad (12)$$

[5]This is the "essentially affine" specification from Duffee (2002), but for a model with full Gaussian dynamics. Also see Cheridito, Filipović, and Kimmel (2007) for further discussion on market price of risk specifications.

[6]References for this standard term structure relationship and others I use subsequently in the article are, for example, Filipović (2009) p. 7 or James and Webber (2000) chapter 3.

2.2 ZLB-GATSM term structures and estimation

K-GATSM forward rates are defined as (see Krippner (2013d) p. 16):

$$\underline{\mathrm{f}}(t,\tau) = \mathrm{f}(t,\tau)\cdot\Phi\left[\frac{\mathrm{f}(t,\tau)}{\omega(\tau)}\right] + \omega(\tau)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{\mathrm{f}(t,\tau)}{\omega(\tau)}\right]^2\right) \tag{13}$$

and $\underline{\mathrm{R}}(t,\tau)$ is obtained using the standard term structure relationship:

$$\underline{\mathrm{R}}(t,\tau) = \frac{1}{\tau}\int_0^\tau \underline{\mathrm{f}}(t,u)\,\mathrm{d}u \tag{14}$$

which is straightforward to evaluate with univariate numerical integration over time to maturity τ. The K-GATSM has already been established as an empirically acceptable approximation to the B-GATSM (which imposes the ZLB in a fully arbitrage-free way), but the relative tractability of the K-GATSM makes it much quicker to apply; see Krippner (2013d), Christensen and Rudebusch (2013a,b), and Wu and Xia (2013).

B-GATSM bond prices may be defined generically as (see Krippner (2013d) p. 6):

$$\underline{\mathrm{P}}^{\mathrm{B}}(t,\tau) = \tilde{\mathbb{E}}_t\left\{\exp\left(-\int_0^\tau \max\{0,\mathrm{r}(t+u)\}\,\mathrm{d}u\right)\right\} \tag{15}$$

and $\underline{\mathrm{R}}^{\mathrm{B}}(t,\tau)$ is obtained using the standard term structure relationship:

$$\underline{\mathrm{R}}^{\mathrm{B}}(t,\tau) = -\frac{1}{\tau}\log\underline{\mathrm{P}}^{\mathrm{B}}(t,\tau) \tag{16}$$

In practice, interest rates for multifactor B-GATSM implementations have been obtained using the multivariate numerical methods of finite-difference grids, interest rate lattices, and Monte Carlo simulations; e.g. see Bomfim (2003), Ueno, Baba, and Sakurai (2006), Ichiue and Ueno (2007), Kim and Singleton (2012), Ichiue and Ueno (2013), Bauer and Rudebusch (2013), and Richard (2013).[7] Recent advances in Priebsch (2013) and Krippner (2013a) offer faster B-GATSM implementations, respectively via a close second-order approximation, and Monte Carlo simulation with a control variate based on the K-GATSM.

Regarding estimation, the state equation for both the K-GATSM and B-GATSM is:

$$x_{t+1} = \theta + \exp(-\kappa\Delta t)(x_t - \theta) + \varepsilon_{t+1} \tag{17}$$

where Δt is the time increment between observations, the subscript t is an integer index for the time series of term structure observations, and ε_{t+1} is the $N\times 1$ vector of innovations to the state variables. The measurement equation for both the K-GATSM and B-GATSM is:

$$\underline{\mathrm{R}}_t = \underline{\mathrm{R}}(x_t,\mathbb{A}) + \eta_t \tag{18}$$

where $\underline{\mathrm{R}}_t$ is the $K\times 1$ vector of interest rate data at time index t, $\underline{\mathrm{R}}(x_t,\mathbb{A})$ is the $K\times 1$ vector of shadow/ZLB-GATSM rates with $\mathbb{A}$ denoting the parameters already noted in

[7] Gorovoi and Linetsky (2004) develops a semi-analytic solution for one factor B-GATSMs, which has been applied in Ichiue and Ueno (2006) and Ueno, Baba, and Sakurai (2006).

section 2.1, and η_t is the $K \times 1$ vector of components unexplained by the shadow/ZLB-GATSM.[8] The K-GATSM uses interest rates defined by equation 14 in equation 18, while the B-GATSM uses interest rates defined by equation 16 in equation 18.

The state space representation has been estimated with the extended, iterated extended, or unscented Kalman filters (e.g. see Kim and Singleton (2012), Krippner (2013d), and Kim and Priebsch (2013) respectively). Christensen and Rudebusch (2013a, b), Krippner (2013d), and Wu and Xia (2013) adopt a common normalization for identification and estimation, which I assume hereafter, of fixed values for b_0, diagonal blocks of real Jordan matrices for $\tilde{\kappa}$ (which allows for repeated eigenvalues), $\tilde{\theta} = 0$, and a lower diagonal matrix σ.[9] As detailed in section 3.1, when one eigenvalue of $\tilde{\kappa}$ and/or κ is set to zero, the further restriction $a_0 = 0$ applies.

3 EMS measures from shadow/ZLB-GATSMs

In this section, I first show how the expected path of the SSR and its long-run expectation (which I interpret as the neutral interest rate) under the risk-adjusted $\mathbb{Q}$ measure can be used to create what I will call the EMS-$\mathbb{Q}$ measure. I use a specification where the non-stationary restriction $\tilde{\kappa}_1 = 0$ is imposed on the first eigenvalue because that results in the most intuitive and parsimonious framework, as I subsequently illustrate in sections 3.2 and 3.3. In section 3.4, I show that the K-ANSM EMS-$\mathbb{Q}$ measures are empirically more robust to different specifications and data sets than SSR estimates, and discuss why EMS-$\mathbb{Q}$ measures should in principle be superior to alternative indicators of the monetary policy stance. Section 3.5 compares EMS-$\mathbb{Q}$ measures in more detail.

EMS-$\mathbb{Q}$ measures can be similarly derived if stationary GATSMs (i.e. where all eigenvalues $\tilde{\kappa}_i$ of $\tilde{\kappa}$ for the shadow GATSM are greater than zero) are used to represent the shadow term structure. However, the interpretation of stationary GATSMs is more involved without adding anything to the principles of the EMS-$\mathbb{Q}$ measures for non-stationary GATSMs, so I relegate those details and illustrations to appendix A.

3.1 EMS-$\mathbb{Q}$ measure with $\tilde{\kappa}_1 = 0$

With the block-diagonal specification and $\tilde{\kappa}_1 = 0$, $x_1(t)$ becomes a Level state variable that follows a random walk. Therefore, the parameter a_0 is restricted to zero, because it can no longer be identified in the estimation, and because it is redundant from an economic perspective given that $x_1(t)$ completely captures the long-run expectation of

[8] The Wu and Xia (2013) K-GATSM is estimated using forward one-month rates constructed from the estimated Svensson (1995)/Nelson and Siegel (1987) model parameters reported in Gürkaynak, Sack, and Wright (2007). Continuous-time K-GATSMs could be estimated analogously using instantaneous forward rates constructed from the same data set. However, I prefer to use an interest rate measurement equation with interest rate data because that is standard in the literature, and forward rate data are not available or readily obtainable for interest rate data sets generated from non-parametric methods; e.g. Bloomberg data (see Kushnir (2009) for method details) and Fama and Bliss (1987).

[9] Different normalizations could be chosen, such as a lower diagonal κ matrix and a diagonal σ matrix, or $\theta = 0$ with $\tilde{\theta}$ estimated, but they would be observationally equivalent representations of the yield curve data. Appendix B shows how non-block-diagonal specifications may be handled, effectively by pre-diagonalizing the original specification.

$\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]$. Specifically, the expected path of the SSR is:

$$\begin{aligned}\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] &= b_0' \exp(-\tilde{\kappa}\tau)\, x(t) \\ &= \left[b_{0,1}, b_{0,L}'\right] \exp\left(-\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\kappa}_L \end{bmatrix}\tau\right)\begin{bmatrix} x_1(t) \\ x_L(t) \end{bmatrix} \\ &= \left[b_{0,1}, b_{0,L}'\right]\begin{bmatrix} 1 & 0 \\ 0 & \exp(-\tilde{\kappa}_L\tau) \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_L(t) \end{bmatrix} \\ &= b_{0,1}x_1(t) + b_{0,L}' \exp(-\tilde{\kappa}_L\tau)\, x_L(t) \end{aligned} \tag{19}$$

where $b_{0,L}$ is the $(N-1)\times 1$ lower-block vector of b_0, $\tilde{\kappa}_L$ is the $(N-1)\times(N-1)$ lower-block matrix of $\tilde{\kappa}$, and $x_L(t)$ is the $(N-1)\times 1$ lower-block vector of $x(t)$. The SSR is:

$$\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]\Big|_{\tau=0} = b_0' x(t) = \mathrm{r}(t) \tag{20}$$

and the long-run expectation/neutral rate is:

$$\lim_{\tau\to\infty} \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] = b_{0,1}x_1(t) \tag{21}$$

A common example of this specification in GATSMs is the three-factor arbitrage-free Nelson and Siegel (1987) model, or hereafter ANSM(3); e.g. see Krippner (2006), Christensen, Diebold, and Rudebusch (2011), and Diebold and Rudebusch (2013). In that case, the remaining two eigenvalues are restricted to be equal, i.e. $\tilde{\kappa}_2 = \tilde{\kappa}_3 > 0$. However, the following expressions would also apply if $\tilde{\kappa}_2$ and $\tilde{\kappa}_3$ were allowed to be distinct, or if just one non-Level factor were used, as with the ANSM(2) used to represent the shadow term structure in section 3.2. Empirically, the restriction $\tilde{\kappa}_1 = 0$ can generally be imposed for parsimony because estimated GATSMs and shadow/ZLB-GATSMs with $\tilde{\kappa}_i > 0$ inevitably turn out to have one eigenvalue $\tilde{\kappa}_1 \gtrsim 0$.[10]

I define the EMS-$\mathbb{Q}$ measure at time t, denoted $\tilde{\xi}(t)$, as the integral of the expected path of the shadow short rate truncated at zero relative to its long-run value, i.e.:

$$\begin{aligned}\tilde{\xi}(t) &= \int_0^\infty \left(b_{0,1}x_1(t) - \max\left\{0, \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]\right\}\right) \mathrm{d}\tau \\ &= \int_0^\infty \left(\max\left\{b_{0,1}x_1(t), b_{0,1}x_1(t) - b_{0,1}x_1(t) - b_{0,L}'\exp(-\tilde{\kappa}_L\tau)\, x_L(t)\right\}\right) \mathrm{d}\tau \\ &= \int_0^\infty \left(\max\left\{b_{0,1}x_1(t), -b_{0,L}'\exp(-\tilde{\kappa}_L\tau)\, x_L(t)\right\}\right) \mathrm{d}\tau \end{aligned} \tag{22}$$

As I will subsequently explain in section 5.2, I intentionally use $\max\left\{0, \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]\right\}$ to define $\tilde{\xi}(t)$, rather than using the non-equivalent alternative $\tilde{\mathbb{E}}_t[\max\{0, \mathrm{r}(t+\tau)\}]$. The latter would result in infinite EMS-$\mathbb{Q}$ measures, which would not correlate empirically with macroeconomic variables and would therefore not be a useful quantity in practice.

3.2 K-ANSM(2) EMS-$\mathbb{Q}$ measure

The EMS-$\mathbb{Q}$ measure in practice is most clearly illustrated for the two-factor K-ANSM, or K-ANSM(2) hereafter. The parsimony of the ANSM(2) when representing the shadow

[10] See, for example, Kim and Singleton (2012) and Ichiue and Ueno (2013) for B-GATSMs, and Wu and Xia (2013) for a K-GATSM example. Krippner (2013c) provides a wide range of results for GATSMs.

term structure makes it very convenient mathematically, and the ANSM(2) is also realistic from a macro-finance perspective for the following four reasons: (1) it contains the Level and Slope components of the term structure, which proxy the first two principal components that in turn explain 99.9 percent of variation in the GSW10 and GSW30 data sets; (2) it approximates the generic GATSM (hence, any GATSM that could be specified) to first order in the sense discussed in Krippner (2013c);[11] (3) the Level and Slope have been shown to relate respectively to inflation and output growth (see Krippner (2008) for discussion of the principles and Diebold, Rudebusch, and Aruoba (2006) for empirical evidence); and (4) related to the previous point, the Level component is modelled as a unit root process. The latter corresponds with empirical evidence that inflation is a strongly persistent process where a unit root typically cannot be rejected; e.g. see Aïssa and Jouini (2003).

For the K-ANSM(2), $\tilde{\mathbb{E}}_t\left[\mathrm{r}\left(t+\tau\right)\right]$ is defined with $b_0' = [1,1]$, and a mean-reversion matrix $\tilde{\kappa}$ with zeros apart from ϕ in the lower diagonal element. Hence, equation 19 becomes:

$$\tilde{\mathbb{E}}_t\left[\mathrm{r}\left(t+\tau\right)\right] = x_1\left(t\right) + x_2\left(t\right)\cdot\exp\left(-\phi\tau\right) \tag{23}$$

with the SSR:

$$\mathrm{r}\left(t\right) = x_1\left(t\right) + x_2\left(t\right) \tag{24}$$

and the long-run expectation:

$$\lim_{\tau\to\infty}\tilde{\mathbb{E}}_t\left[\mathrm{r}\left(t+\tau\right)\right] = x_1\left(t\right) \tag{25}$$

Rather than immediately deriving a general analytic expression for $\tilde{\xi}\left(t\right)$ at this stage, it is more insightful to illustrate the EMS-$\mathbb{Q}$ measure graphically, and so figures 3 and 4 provide examples of the two cases that can occur in practice.[12] Note that, unless stated otherwise, all figures in section 3 are based on the K-ANSM(2) and K-ANSM(3) models from Krippner (2013d) which are estimated (with the iterated extended Kalman filter) using month-end Gürkaynak, Sack, and Wright (2007) yield curve data with maturities out to 30 years (hereafter the GSW30 data set). Specifically, the maturities of the GSW30 data set are 0.25, 0.5, 1, 2, 3, 4, 5, 7, 10, 15, 20, and 30 years, the sample period is November 1985 (the first month from which 30 year data were available) to July 2013 (the last month available at the time of the Krippner (2013d) estimations). I also use results estimated with GSW10 data (i.e. the GSW30 data set without the last three yields), but the GSW30 data should in principle provide better estimates of the Level state variable and hence the EMS-$\mathbb{Q}$ measure, and section 3.4 shows that is indeed the case in practice. Note also that the principles of the illustrations also apply to B-GATSMs, but the estimates of the shadow-GATSM parameters and state variables from the data would differ somewhat.

[11] Krippner (2013c) sets the near-zero eigenvalues of the mean-reversion matrix $\tilde{\kappa}$ for the generic GATSM equal to zero and the non-persistent eigenvalues to their mean value denoted as ϕ. The components of the generic GATSM then condense to the ANSM(2). The extension to the ANSM(3) allows an additional term in the Taylor expansion for the non-persistent eigenvalues around ϕ, which results in the addition of the Bow component relative to the ANSM(2).

[12] Other possibilities with $x_1\left(t\right) < 0$ are purely mathematical; such estimates do not arise in practice, and a negative neutral interest rate would lack an economic interpretation in any case.

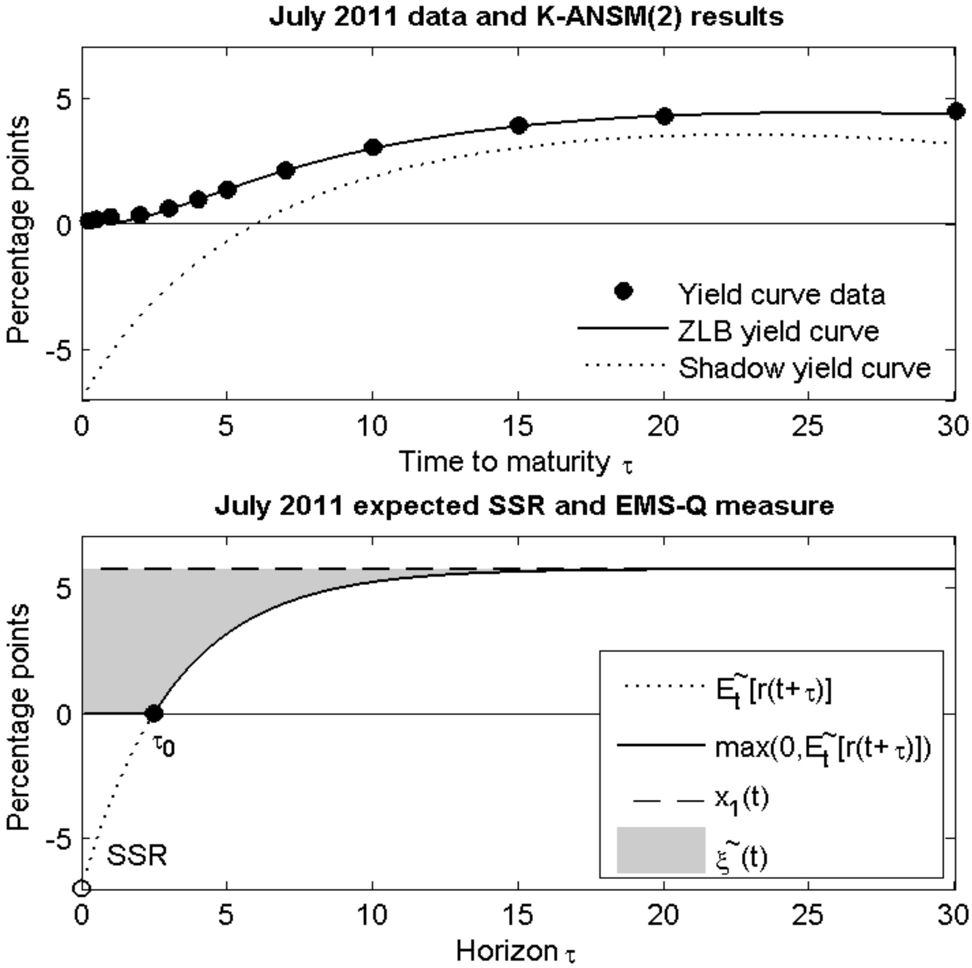

Figure 3: U.S. yield curve data, estimated K-ANSM(2) results, and the EMS-$\mathbb{Q}$ measure for July 2011. This example illustrates the case for a negative SSR, which in turn corresponds to an unconventional/ZLB monetary policy environment.

Figure 3 illustrates $\tilde{\xi}(t)$ as the shaded region for an unconventional/ZLB environment. This example is as at July 2011, and I use that date often throughout the article to demonstrate alternative models. For figure 3, the estimated Level and Slope state variables are $x_1(t) = 5.70\%$ and $x_2(t) = -12.62\%$, giving an SSR of $\mathrm{r}(t) = -6.91\%$, which means the truncation $\max\left\{0, \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]\right\}$ will bind for future horizons $\tau \in [0, \tau_0]$. The value of τ_0 is obtained by setting $\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau_0)] = x_1(t) + x_2(t) \cdot \exp(-\phi\tau_0) = 0$ and solving for τ_0, with the result:

$$\tau_0 = -\frac{1}{\phi} \log\left[-\frac{x_1(t)}{x_2(t)}\right] \tag{26}$$

which has a value of $\tau_0 = 2.32$ years in figure 3, given the parameter estimate $\phi = 0.3196$.

In general, $\tilde{\xi}(t)$ will have two components when $\mathrm{r}(t) < 0$. Specifically, equation 22 with $b_{0,L} = 1$ and $\tilde{\kappa}_L = \phi$ gives:

$$\begin{aligned} \tilde{\xi}(t) &= \int_0^\infty \left(\max\left\{ x_1(t), -x_2(t) \cdot \exp(-\tilde{\kappa}\tau) \right\} \right) \mathrm{d}\tau \\ &= \int_0^{\tau_0} x_1(t)\, \mathrm{d}\tau - \int_{\tau_0}^\infty x_2(t) \cdot \exp(-\phi\tau)\, \mathrm{d}\tau \\ &= x_1(t) \cdot \tau_0 - x_2(t) \cdot \frac{1}{\phi} \exp(-\phi\tau_0) \end{aligned} \quad (27)$$

Evaluating those integrals as at July 2011 and summing them gives $\tilde{\xi}(t) = 31.47$ percent.[13]

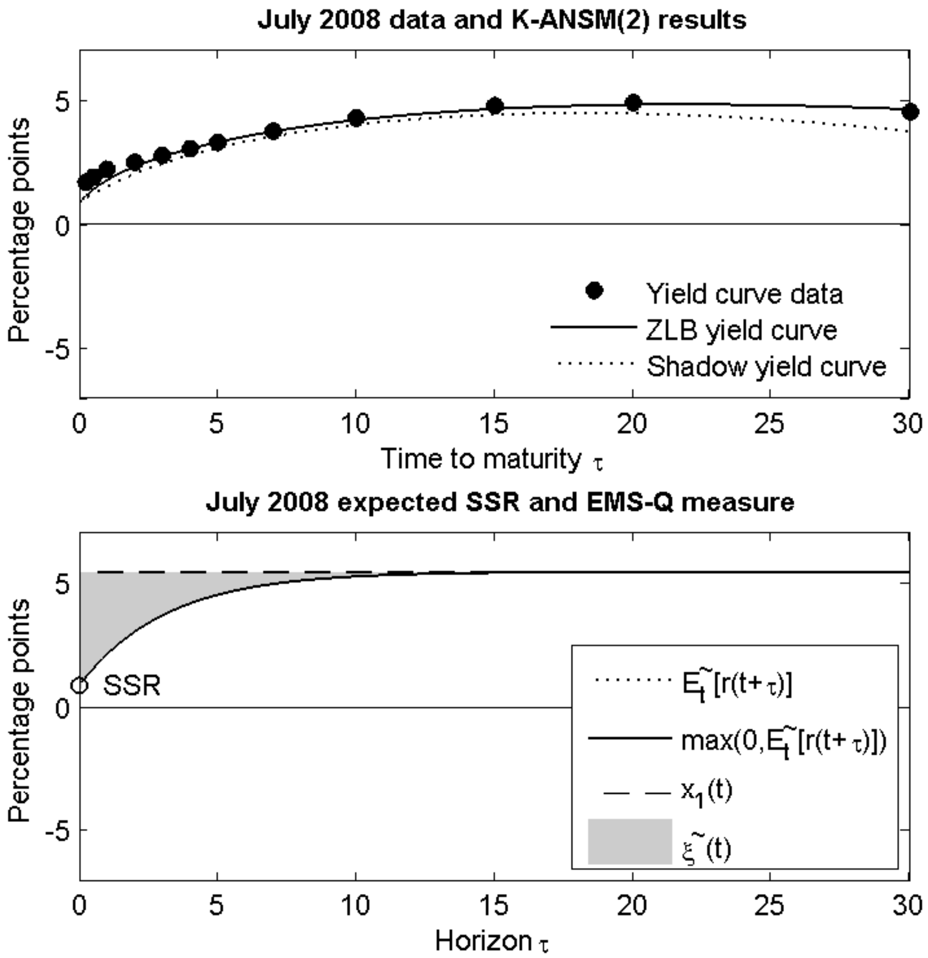

Figure 4: U.S. yield curve data, estimated K-ANSM(2) results, and the EMS-$\mathbb{Q}$ measure for July 2008. This example illustrates the case for a positive SSR, which in turn corresponds to a conventional/non-ZLB monetary policy environment.

Figure 4 illustrates the economic stimulus $\tilde{\xi}(t)$ for July 2008, which is an example of a conventional/non-ZLB environment. In this case the estimated Level and Slope state variables are $x_1(t) = 5.41\%$ and $x_2(t) = -4.54\%$, giving an SSR of 0.87%, and so the truncation $\max\left\{0, \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]\right\}$ will not bind for any future horizons.

[13] Interest rates are expressed as an annualized percent per year. Hence, the integral has the units of percent per year multiplied by years, which produces the unit of percent.

In general, $\tilde{\xi}(t)$ will have just a single component when $\mathrm{r}(t) \geq 0$, i.e.:

$$\begin{aligned} \tilde{\xi}(t) &= -\int_0^\infty x_2(t) \cdot \exp(-\phi\tau)\,\mathrm{d}\tau \qquad (28) \\ &= -x_2(t) \cdot \frac{1}{\phi} \qquad (29) \end{aligned}$$

The result of that integral as at July 2008 is $\tilde{\xi}(t) = 13.79$. Note that $\tilde{\xi}(t)$ can and has taken on negative values (e.g. see figures 6 and 7 subsequently), which occurs if $x_2(t) > 0$. That condition has an economic interpretation of a restrictive stance of monetary policy; specifically, $x_2(t) > 0$ corresponds to $\mathrm{r}(t) > \lim_{\tau\to\infty} \mathbb{E}_t[\mathrm{r}(t+\tau)]$, hence $\underline{\mathrm{r}}(t) - \lim_{\tau\to\infty} \mathbb{E}_t[\mathrm{r}(t+\tau)] > 0$, which in turn implies that the current actual interest rates are restrictive relative to their neutral interest rate.

Summarizing the two cases, the general analytic expression $\tilde{\xi}(t)$ with the ANSM(2) is therefore:

$$\tilde{\xi}(t) = \begin{cases} x_1(t) \cdot \tau_0 - x_2(t) \cdot \frac{1}{\phi} \exp(-\phi\tau_0) & \text{if } \mathrm{r}(t) < 0 \\ -x_2(t) \cdot \frac{1}{\phi} & \text{if } \mathrm{r}(t) \geq 0 \end{cases} \qquad (30)$$

3.3 K-ANSM(3) EMS measure

For the K-ANSM(3), $\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]$ is defined with $b_0' = [1, 1, 0]$ and a mean-reversion matrix $\tilde{\kappa}$ with zeros apart from the following lower diagonal Jordan block:

$$\tilde{\kappa}_L = \begin{bmatrix} \phi & -\phi \\ 0 & \phi \end{bmatrix} \qquad (31)$$

Hence, equation 19 becomes:

$$\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] = x_1(t) + x_2(t) \cdot \exp(-\phi\tau) + x_3(t) \cdot \phi\tau \exp(-\phi\tau) \qquad (32)$$

which is the ANSM(2) expression with the addition of a Bow component, i.e. $x_3(t) \cdot \phi\tau \exp(-\phi\tau)$,[14] that improves the fit of mid-maturity rates. The expressions for the long-run expectation and the SSR are the same as for the K-ANSM(2).

In principle, the general analytic expression for $\tilde{\xi}(t)$ from the ANSM(3) would involve the two cases for the ANSM(2) and an additional potential case where $\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] < 0$ for intermediate values of τ (i.e. $0 < \tau_1 < \tau < \tau_2$). However, the latter case has never occurred in practice, and so the $\tilde{\xi}(t)$ general analytic expression for the ANSM(3) is analogous to the ANSM(2) with the addition of the Bow component, i.e.:

$$\tilde{\xi}(t) = \begin{cases} \begin{array}{c} x_1(t) \cdot \tau_0 - x_2(t) \cdot \frac{1}{\phi} \exp(-\phi\tau_0) \\ -x_3(t) \cdot \left[\left(\frac{1}{\phi} + \tau_0\right) \exp(-\phi\tau_0)\right] \end{array} & \text{if } \mathrm{r}(t) < 0 \\ -S(t) \cdot \frac{1}{\phi} - B(t) \cdot \frac{1}{\phi} & \text{if } \mathrm{r}(t) \geq 0 \end{cases} \qquad (33)$$

[14]See Diebold and Rudebusch (2013) for further discussion on Nelson and Siegel (1987) models and ANSMs. My preferred name "Bow" is often referred to as "Curvature" in that book and the related literature. However, the Slope component itself has a natural curvature, resulting from its exponential decay functional form, so Bow is a less ambiguous (and syllable-saving) name for the third ANSM component.

where τ_0 is readily found numerically (e.g. using the "fzero" function in MatLab) as the solution to $\tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau_0)] = x_1(t) + x_2(t) \cdot \exp(-\phi\tau_0) + x_3(t) \cdot \phi\tau \exp(-\phi\tau) = 0$. In figure 5, $\tau_0 = 2.38$ years, which compares to $\tau_0 = 2.32$ years for the ANSM(2).

Figure 5 illustrates $\tilde{\xi}(t)$ for the ANSM(3) as at July 2011. In this case the estimated Level, Slope, and Bow state variables of $x_1(t) = 6.11\%$, $x_2(t) = -8.82\%$, and $x_3(t) = -8.46\%$ give $\tilde{\xi}(t) = 33.87$, which is close to the value of $\tilde{\xi}(t) = 31.47$ for the K-ANSM(2) in figure 3. However, the SSR of $\mathrm{r}(t) = -2.71\%$ is distinctly different from the value of -6.91% for the ANSM(2) in figure 3, which in turn arises from the influence of the Bow component in fitting the observed yield curve data.

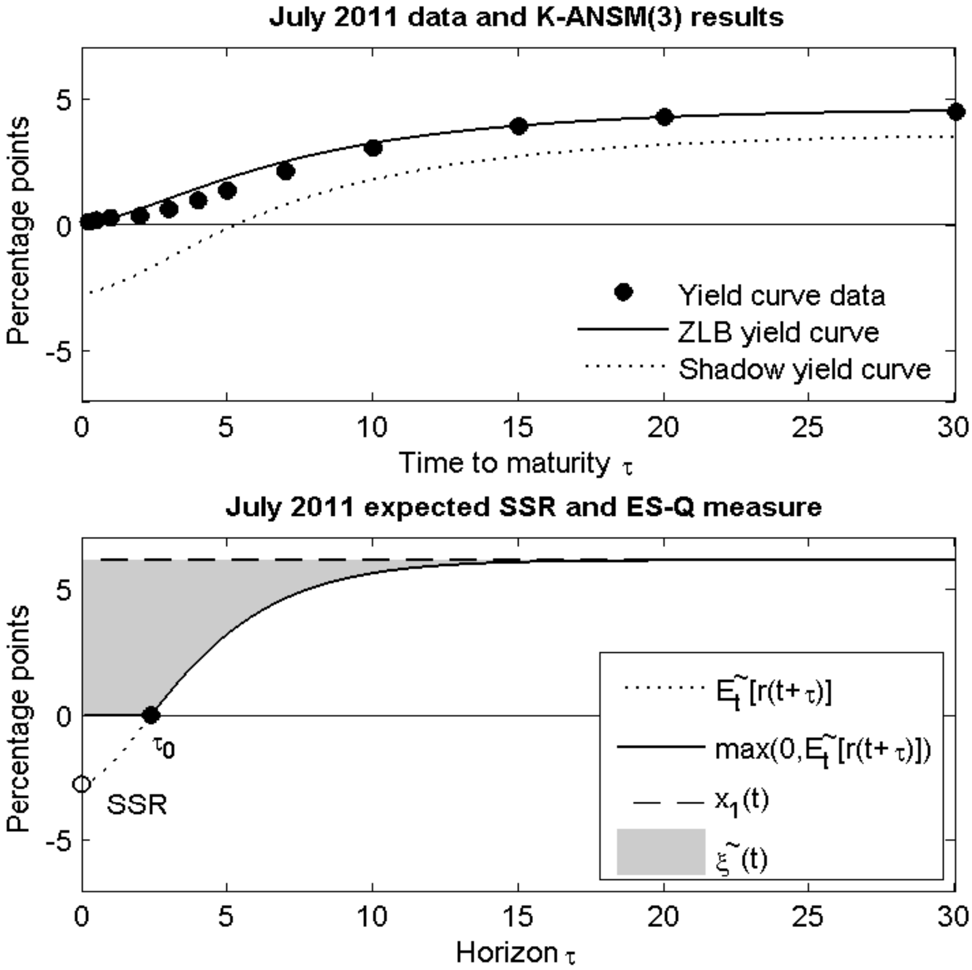

Figure 5: U.S. yield curve data, estimated K-ANSM(3) results, and the EMS-$\mathbb{Q}$ measure for July 2011. This example illustrates an alternative estimate for the unconventional/ZLB monetary policy environment illustrated in figure 3. The resulting EMS measure is similar to figure 3, while the SSR is distinctly different.

3.4 Comparing K-ANSM EMS measures to SSRs

3.4.1 Empirical perspective

Figure 6 plots the time series of the EMS-$\mathbb{Q}$ measures and the SSRs for the K-ANSM(2) and K-ANSM(3) results obtained with the GSW30 data set, and figure 7 plots the results obtained with the GSW10 data set. Note that I have indicated NBER recessions with the shaded areas, as I will do for all full-sample time series figures, to provide a very preliminary gauge on how the series relate to real output growth. Section 5.1 discusses the more systematic analysis that would obviously be required to fully assess the practical use of EMS-$\mathbb{Q}$ measures.

One key point from the time series figures is that, within and across both figures, the EMS-$\mathbb{Q}$ measures are much closer to each other than the SSR estimates. This result suggests that the EMS-$\mathbb{Q}$ measures are more robust than SSRs with respect to the number of factors used to specify the model and the data used for estimation. That said, there is still some variation between the EMS-$\mathbb{Q}$ measures, particularly a notable divergence at the end of the sample, which warrants further discussion in section 3.5.

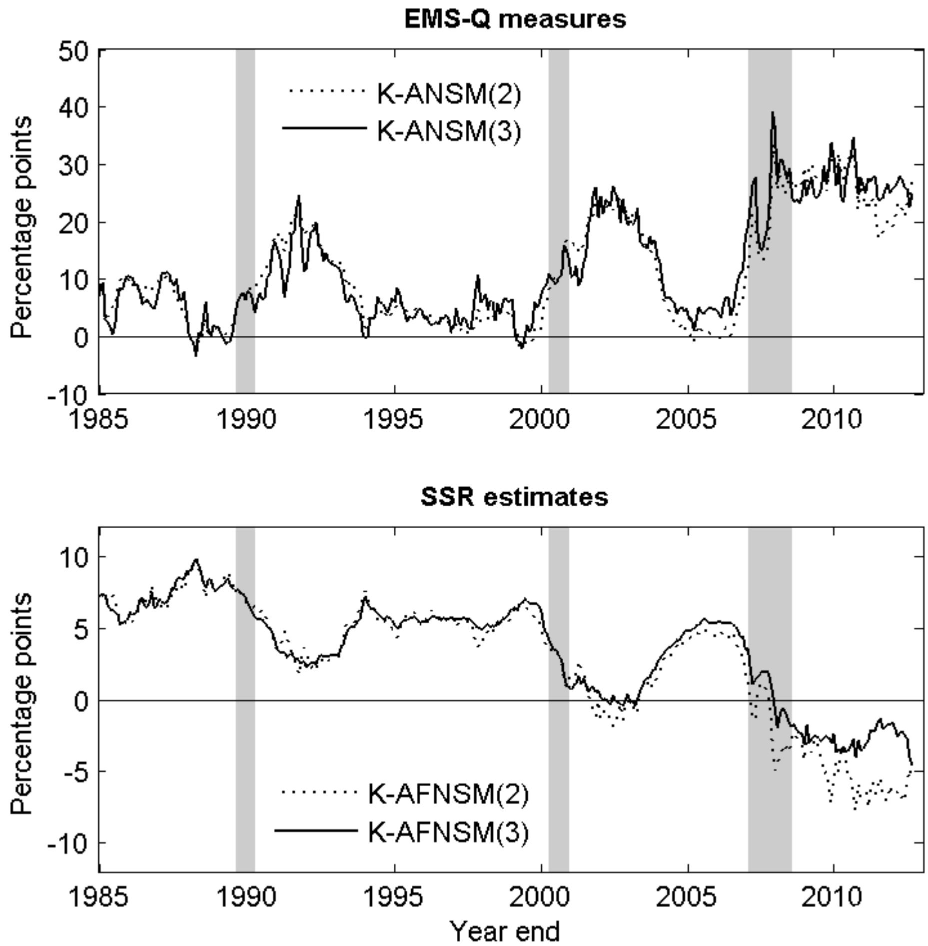

Figure 6: Time series plots of the EMS-$\mathbb{Q}$ measures and the SSRs for the K-ANSM(2) and K-ANSM(3) estimated with GSW30 data. The EMS-$\mathbb{Q}$ measures are more robust across the estimated models than the SSR estimates.

Another key point from figures 6 and 7 is that EMS-$\mathbb{Q}$ measures appear to correlate well with output growth. Specifically, the periods where the EMS-$\mathbb{Q}$ measures are high follow the onset of NBER recessions, and the larger recession associated with the Global Financial Crisis is followed with a more extreme and prolonged EMS-$\mathbb{Q}$ measure than the previous two recessions. The higher EMS-$\mathbb{Q}$ measures are in turn consistent with easier monetary policy to close the output gaps that arise in the wake of the recessions. In that regard, the SSR estimates have a complementary profile, but with wider variation between different model estimates as previously mentioned.

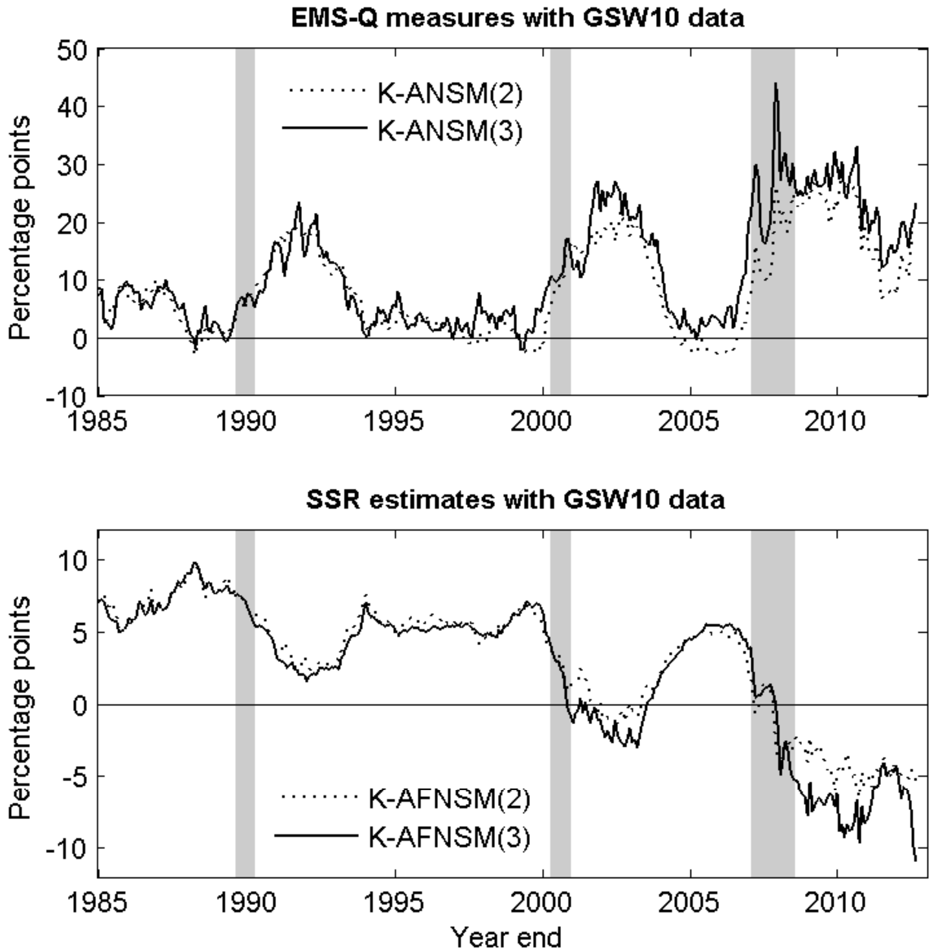

Figure 7: Time series plots of the EMS-$\mathbb{Q}$ measures and the SSRs for the K-ANSM(2) and K-ANSM(3) estimated with GSW10 data. The EMS-$\mathbb{Q}$ measures are more robust across the estimated models than the SSR estimates.

3.4.2 Theoretical perspective

To highlight the economic interpretation of the EMS-$\mathbb{Q}$ measures, note that they change continuously with the changing shape of the yield curve, which in turn implies: (1) a change to the expected path of the SSR; and/or (2) a change to the long-run SSR expectation, which is used as a proxy for the neutral interest rate (and both are coincident with the long-run expectation of the actual short rate). I discuss each of the two components in turn.

In non-ZLB periods, the expected path of the SSR can change independently of changes to the policy rate, which is appropriate because even conventional monetary policy operates partly through signalling and expectations; e.g. see Walsh (2003) chapter 10 for discussion of the principles and Gürkaynak, Sack, and Swanson (2005) for empirical evidence. In ZLB periods, forward guidance and expectations are an important component of unconventional monetary policy; e.g. see Woodford (2012) section 1. Note, however, that $\tilde{\mathbb{E}}_t [\mathrm{r}(t+\tau)]$ can and does change beyond the direct influence of either conventional or unconventional monetary policy actions, essentially by any factors that influence the yield curve. Therefore, $\tilde{\mathbb{E}}_t [\mathrm{r}(t+\tau)]$ and EMS-$\mathbb{Q}$ measures should be treated as market expectations variables subject to central bank influence, rather than quantities explicitly controlled by the central bank like the FFR or asset purchase programs. In that respect, $\tilde{\mathbb{E}}_t [\mathrm{r}(t+\tau)]$ is similar to the lift-off horizon, which can be influenced but not explicitly controlled by the central bank.

Regarding the long-run SSR expectation/neutral interest rate, it can change to reflect changes in expected macroeconomic fundamentals (such as long-run inflation expectations and potential output growth) in both non-ZLB and ZLB environments; Krippner (2008) contains related discussion. Potential output growth is generally considered to be beyond the influence of the central bank, while the policy goals (such as an inflation target) and the credibility of the central bank may influence long-run inflation expectations.

Because EMS-$\mathbb{Q}$ measures account for the path of the expected actual short rate relative to its long-run expectation, they should provide a more comprehensive summary of the stimulus from interest rates and the yield curve compared to any single actual or estimated interest rate, or interest rate spreads. This observation applies even in non-ZLB/conventional monetary policy environments, where identical settings of the FFR with different expectations of future movements imply different degrees of monetary policy stimulus. For example, the FFR was cut to 1.00% on 25 June 2003 where it remained until a hike to 1.25% on 10 August 2004. However, yield curve data at the beginning of that period was distinctly lower than at the end. Figures 6 and 7 indicate that difference with higher EMS-$\mathbb{Q}$ measures, and figure 8 indicate it with lower SSR estimates (which are distinctly lower than the plotted 3 month rate, and the FFR at the time).

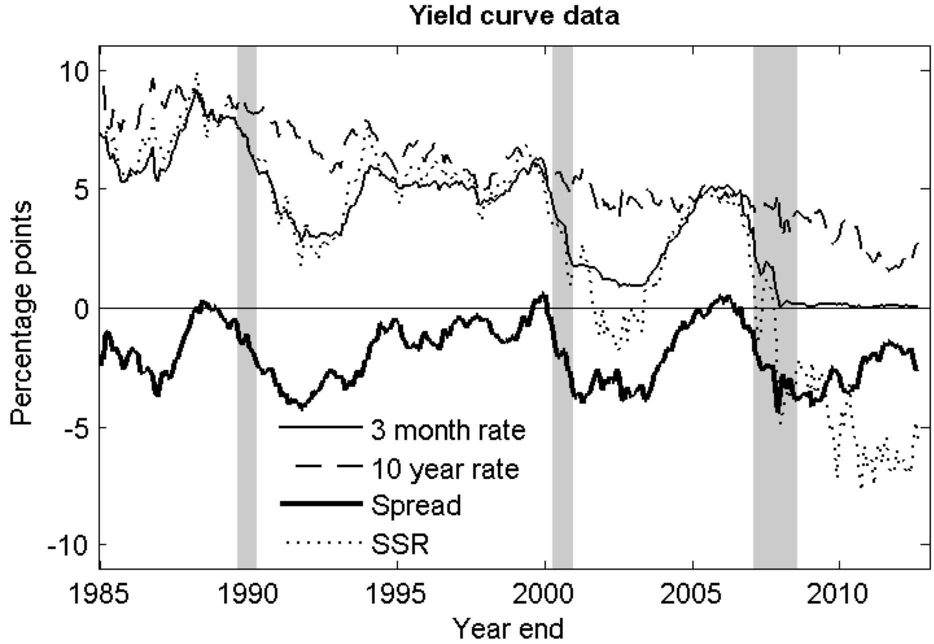

Figure 8: Actual interest rates and their spread (short- less long-maturity rate), and the K-ANSM(2) SSR. At the ZLB, the 3 month rate is uninformative, the spread is directionally misleading, and the SSR overstates the degree of monetary policy stimulus.

More generally, if the FFR or short-maturity rate is not compared to a neutral rate, then it could be misleading as a metric for monetary policy. At the very least, the FFR should be adjusted for inflation, based on proxies for inflation expectations (such as historic inflation and/or surveys). Conversely, EMS-$\mathbb{Q}$ measures directly deliver a quantity adjusted for inflation expectations inherent in the Level component of the K-ANSM (or B-GATSM), which in turn reflects interest rates for longer maturities. In non-ZLB/conventional monetary policy environments, using the spread between the interest rates of two maturities on the yield curve could be used to mitigate the issues associated with using a single interest rate as a metric for monetary policy stimulus. However, EMS-$\mathbb{Q}$ measures remain superior, in principle, to any spread because EMS-$\mathbb{Q}$ measures account for the different paths of the actual short rate that might underlie any particular spread.

In ZLB/unconventional monetary policy environments, EMS-$\mathbb{Q}$ measures have very distinct advantages relative to actual interest rates, spreads between actual interest rates, and SSRs. The advantage over actual interest rates is that EMS-$\mathbb{Q}$ measures can continue to reflect unconventional policy easing, while the ZLB attenuates further downward interest rates movements; in particular, short-maturity interest rates essentially remain static at or near zero and so cannot reflect further policy easing. The advantage over actual interest rate spreads is even more pronounced. Specifically, figure 8 plots the 3-month less 10-year spread, which is a standard indicator of the yield curve slope (albeit inverted here to correspond with the profile of the 3-month rate and the SSR). Note that periods of tight (easy) policy in non-ZLB/conventional monetary policy environments have inevitably corresponded with high (low) values of the spread. However, in the ZLB/unconventional monetary policy environment since December 2008, the spread steadily rises (apart from tapering talk at the end of the sample) because the 10-year rate falls while the 3-month rate remains at approximately zero. Therefore, the rising spread could be misinterpreted as a tightening of monetary policy, when in reality it was eased substantially via unconventional methods. The EMS-$\mathbb{Q}$ measures to move in the direction of easing, and so are consistent with unconventional monetary policy events.

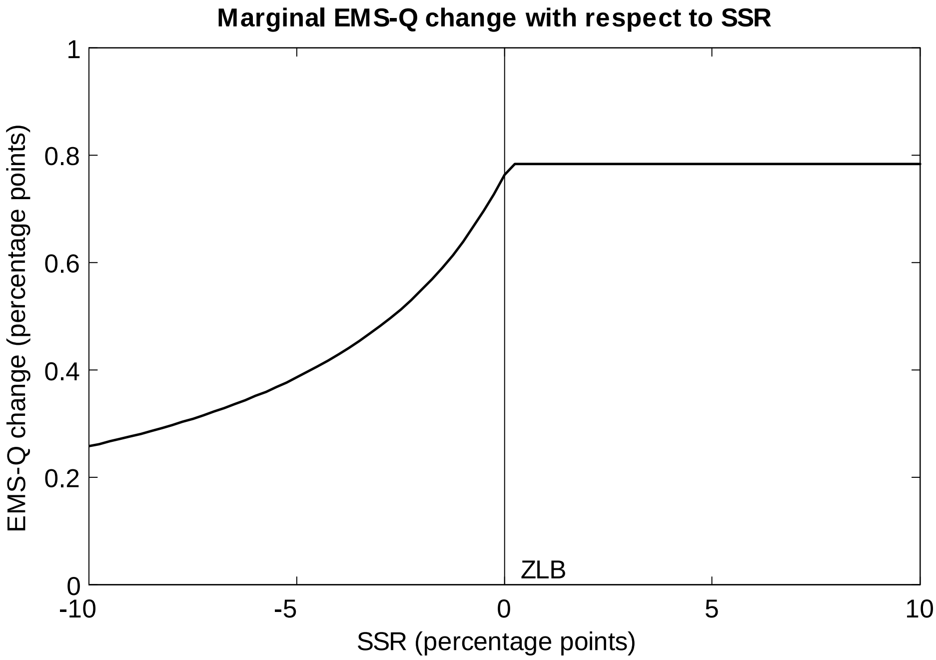

Figure 9: The change in the K-ANSM(2) EMS-$\mathbb{Q}$ measure for 25 bp decreases in the SSR as a function of the starting values of the SSR on the x axis. The monetary policy stimulus from decreasing the SSR (i.e. moving from right to left on the x axix) is attenuated by the ZLB as the SSR moves through the ZLB to more negative values.

While movements in the SSR are also qualitatively consistent with actual monetary policy events, a literal interpretation of the SSR profile is that a fall by a given amount offers a similar amount of policy stimulus regardless of the SSR level. Specifically, that is how the SSR would effectively be treated in any econometric analysis. However, as initially discussed in the introduction and illustrated in figure 2, it is actual interest rates constrained by the ZLB rather than SSRs that are faced by economic agents. Hence, a decline in the SSR when it is positive is associated with similar-sized falls in short-maturity interest rates and corresponding falls in interest rates along the yield curve. Conversely, when the SSR is zero or negative, the interest rates for short maturities cannot fall as the SSR declines to more negative values, and falls in interest rates along the yield curve are also attenuated. This attenuation becomes more pronounced as the SSR becomes more negative, essentially because forward rates for increasingly long horizons will already be subject to the ZLB constraint.

Figure 9 illustrates the attenuation effect by plotting the increase in the K-ANSM(2) EMS-$\mathbb{Q}$ measure for a 25 basis point (bp) decrease in the SSR, as a function of the starting values of the SSR (e.g. what is the EMS-$\mathbb{Q}$ measure change when the SSR is lowered from 10 to 9.75%, 9.75 to 9.50% etc.). Note that the Level variable $x_1(t)$ remains constant at 5% throughout (hence the long-run SSR expectation/neutral interest rate is 5%) and the SSR is varied by changing the Slope state variable $x_2(t)$. The increases in the EMS-$\mathbb{Q}$ measure are essentially identical for each 25 bp cut in the SSR from 10 to 0% (i.e. moving from right to left on the x axis). However, from 0 to -10%, the increases in the EMS-$\mathbb{Q}$ measure for each successive 25 bp cut in the SSR become lower. This pattern shows how the notional monetary policy stimulus from lower SSRs gets increasingly attenuated as the SSR moves through the ZLB to more negative values. In other words, the monetary policy stimulus from the SSR becomes non-linear at and below SSR values of zero. The EMS-$\mathbb{Q}$ measure explicitly accounts for that non-linearity, and so should in principle provide a better measure of monetary policy stimulus.

3.5 Comparing K-ANSM EMS-$\mathbb{Q}$ measures

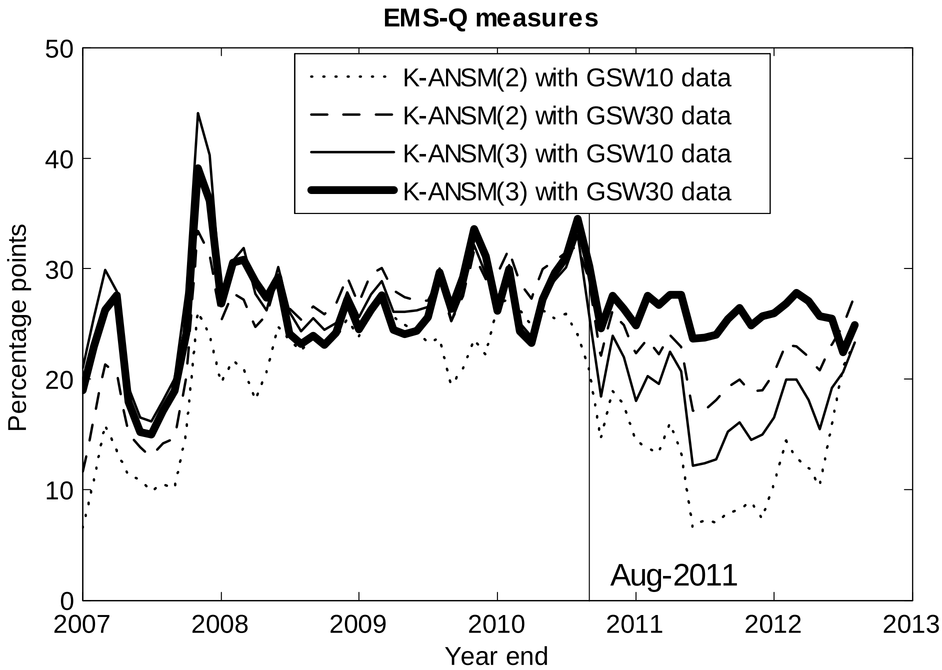

Figure 10: EMS-$\mathbb{Q}$ measures for the K-ANSMs based on GSW10 and GSW30 data. The K-ANSM(3) GSW30 EMS-$\mathbb{Q}$ measure is "best", as discussed in the text, while the other measures show some divergence from August 2011.

While the EMS-$\mathbb{Q}$ measures are apparently more robust than SSRs, the variation between them is sufficient to ask the question: which EMS-$\mathbb{Q}$ measure is "best"? Based on the results and associated discussion below, the K-ANSM(3) GSW30 EMS-$\mathbb{Q}$ measure appears to be better than the other EMS-$\mathbb{Q}$ measures, but all measures can likely be improved on as I later discuss in section 5.1. Figure 10 replots all of the EMS measures from figures 7 and 8 over the latter part of sample so the main divergences noted in the previous section are easier to see.

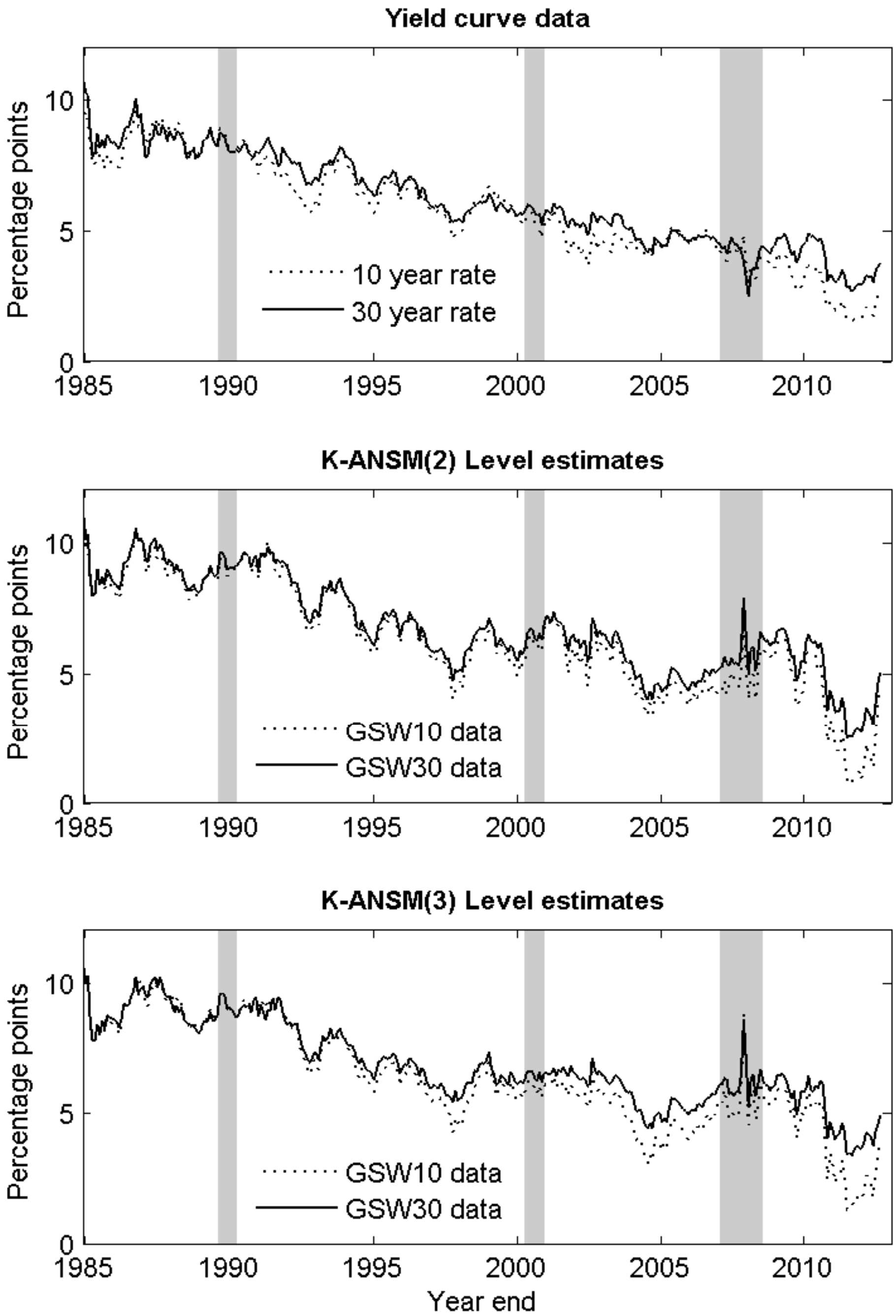

Figure 11: 10 and 30 year maturity interest rates, and the Level state variables for the K-ANSM(2) and K-ANSM(3) estimated using the GSW10 and GSW30 data sets. The more pronounced declines in the 10 year rates lead to lower GSW10 Level estimates at the end of the sample.

The most notable divergences occur from August 2011 when the ANSM(3) GSW30 EMS-$\mathbb{Q}$ measure remains relatively steady while the other EMS-$\mathbb{Q}$ measures move to less positive values. The latter movements are consistent with a policy tightening and/or market anticipation of more restrictive interest rates relative to the neutral rate, but that does not accord with the Federal Reserve's policy guidance at the time. Indeed, August 2011 was the month in which the Federal Reserve's first introduced explicit conditional forward guidance into its post-meeting press release, and that was generally regarded as an easing event by the market. Specifically, the 10 year rate fell by 68 bps from July to August 2011, and later declined by a further 86 bps to a low of 1.55 percent in July 2012.

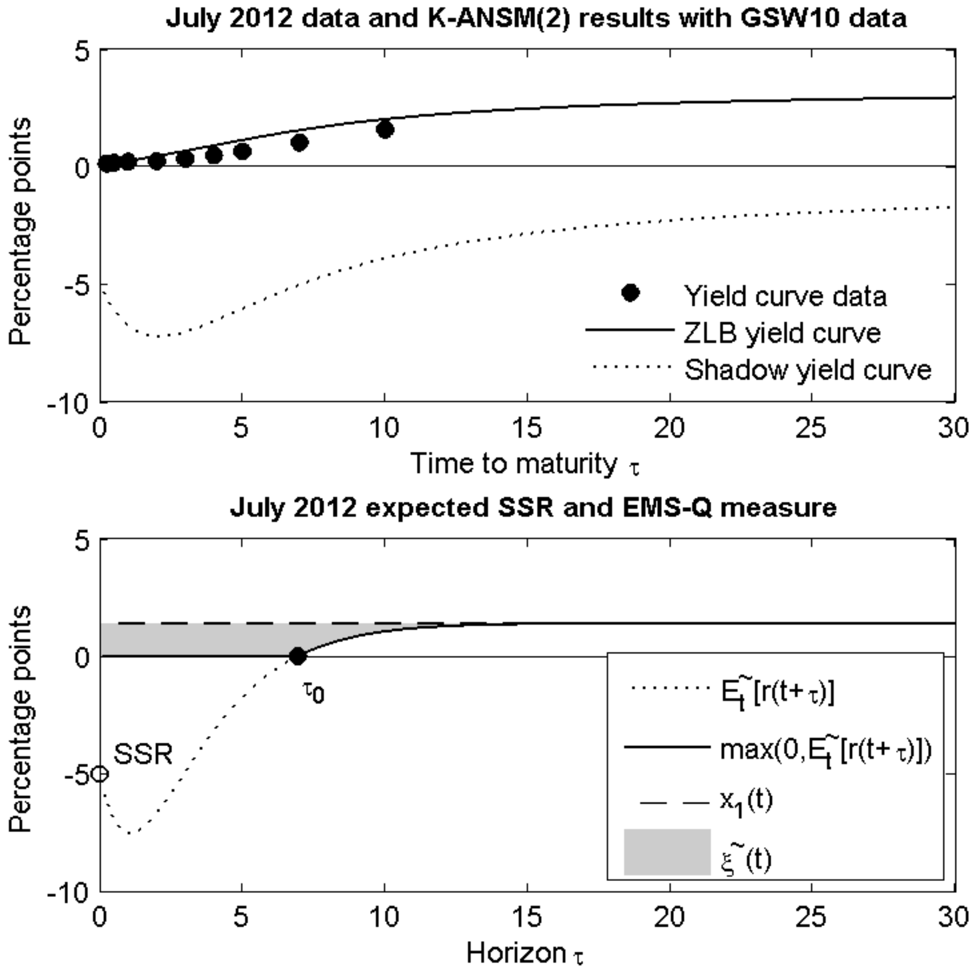

Figure 12: U.S. yield curve data, estimated K-ANSM(2) results based on GSW10 data, and the EMS measure for July 2012. This illustrates how a low estimate of the Level state variable attenuates the EMS measure.

The anomalous movement in all but the ANSM(3) GSW30 EMS measure can be explained by the influence of the Level state variable on the EMS-$\mathbb{Q}$ measure, which is in turn largely due to the dependence of the Level estimate on the data used for estimation. Essentially, the EMS-$\mathbb{Q}$ measure is attenuated to the extent that cyclical variation in longer-maturity yield curve data gets translated into the Level estimate. To illustrate that mechanism, figure 11 plots the GSW 10 and 30 year maturity interest rates along with the K-ANSM Level state variable estimates based on the GSW10 and GSW30 data sets. The 10 year interest rate data has greater cyclical variation than the 30 year interest rate data, most notably showing a more pronounced decline since 2011. That greater cyclical variation translates into a greater cyclical variation of the Level estimate from the GSW10 data compared to the Level estimates from the GSW30 data; in particular, the decline in the GSW10 Level estimates are more pronounced since 2011.

In turn, as illustrated in figure 12, the lower GSW10 Level estimates are treated as a lower long-run SSR expectation/neutral interest rate, which severely attenuates the area between $\tilde{\mathbb{E}}_t\left[\mathrm{r}\left(t+\tau\right)\right]$ and $x_1\left(t\right)$, and leads to a low EMS-$\mathbb{Q}$ measure. Conversely, GSW30

Level estimates show less cyclical variation, which translates into less attenuation and therefore variation in the associated EMS-$\mathbb{Q}$ measures.[15]

In essence then, the longer-maturity data in the GSW30 data set appears to provide a better empirical anchor for long-run SSR expectations that are used as a proxy for the neutral interest rate. In turn, that is consistent with the principle that interest rate data spanning longer horizons should be less influenced by prevailing monetary policy and economic cycles, and more influenced by the long-run macroeconomic fundamentals of potential growth and inflation expectations mentioned in section 3.4.

Regarding the two EMS-$\mathbb{Q}$ measures based on GSW30 data, the K-ANSM(3) EMS-$\mathbb{Q}$ measure should in principle be better than the K-ANSM(2) EMS-$\mathbb{Q}$ measure. The K-ANSM(3) has the additional flexibility of the Bow component to better explain cyclical movements in short- and medium-maturity rates, leaving the Level less influenced by cyclical changes in those rates. Conversely, the K-ANSM(2) is forced to accommodate some of the cyclical movements in medium-maturity rates within the Level estimate, and that will lead to more attenuation in the associated EMS-$\mathbb{Q}$ measure relative to the K-ANSM(3).

4 EMS-$\mathbb{P}$ measures

EMS-$\mathbb{P}$ measures for shadow/ZLB-GATSMs may be defined analogous to EMS-$\mathbb{Q}$ measures. Specifying both $\tilde{\kappa}$ and κ to be block diagonal with their first eigenvalues restricted to zero, as with the supplementary two-factor K-GATSM I present in this section, again results in most intuitive and parsimonious framework. Appendix B illustrates how both of those aspects could be relaxed.

In the example that follows, I have specified and estimated (with GSW30 data and the iterated extended Kalman filter) a two-factor K-GATSM with $\kappa = \text{diag}[0, \kappa_2]$ and $\tilde{\kappa} = \text{diag}[0, \tilde{\kappa}_2]$. This specification fulfills the intended restrictions $\kappa_1 = 0$ and $\tilde{\kappa}_1 = 0$ while also resulting in straightforward functional forms for illustrative purposes. That is, with $a_0 = 0$ and $b_0' = [1, 1]$, equation 5 becomes:

$$\begin{aligned} \mathbb{E}_t\left[\mathrm{r}\left(t+\tau\right)\right] &= \theta_1 + \theta_2 + x_1\left(t\right) - \theta_1 + \left[x_2\left(t\right) - \theta_2\right] \cdot \exp\left(-\kappa_2 \tau\right) \\ &= \theta_2 + x_1\left(t\right) + \left[x_2\left(t\right) - \theta_2\right] \cdot \exp\left(-\kappa_2 \tau\right) \end{aligned} \tag{34}$$

with the SSR:

$$\mathrm{r}\left(t\right) = x_1\left(t\right) + x_2\left(t\right) \tag{35}$$

the long-run expectation:

$$\lim_{\tau \to \infty} \mathbb{E}_t\left[\mathrm{r}\left(t+\tau\right)\right] = \theta_2 + x_1\left(t\right) \tag{36}$$

The EMS-$\mathbb{Q}$ measure is as already defined in section 3.1, and the EMS-$\mathbb{P}$ measure is:

$$\xi\left(t\right) = \begin{cases} \int_0^{\tau_0} \theta_2 + x_1\left(t\right)\,\mathrm{d}\tau - \int_{\tau_0}^{\infty} \left[x_2\left(t\right) - \theta_2\right] \cdot \exp\left(-\kappa_2 \tau\right)\,\mathrm{d}\tau & \text{if } \mathrm{r}\left(t\right) < 0 \\ -\int_{\tau_0}^{\infty} \left[x_2\left(t\right) - \theta_2\right] \cdot \exp\left(-\kappa_2 \tau\right)\,\mathrm{d}\tau & \text{if } \mathrm{r}\left(t\right) \geq 0 \end{cases} \tag{37}$$

$$= \begin{cases} \left[\theta_2 + x_1\left(t\right)\right] \cdot \tau_0 - \left[x_2\left(t\right) - \theta_2\right] \cdot \frac{1}{\kappa_2} \exp\left(-\kappa_2 \tau_0\right) & \text{if } \mathrm{r}\left(t\right) < 0 \\ -\left[x_2\left(t\right) - \theta_2\right] \cdot \frac{1}{\kappa_2} & \text{if } \mathrm{r}\left(t\right) \geq 0 \end{cases} \tag{38}$$

[15] The value of $\tau_0 = 6.95$ years based on the GSW10 data in figure 8 is also relatively large compared to the value of $\tau_0 = 3.96$ years based on the GSW30 data.

Figure 13 plots an example of the EMS-$\mathbb{Q}$ and EMS-$\mathbb{P}$ measures for July 2011. Figure 14 plots the time series of EMS-$\mathbb{Q}$ and EMS-$\mathbb{P}$ measures, and also the associated SSR.

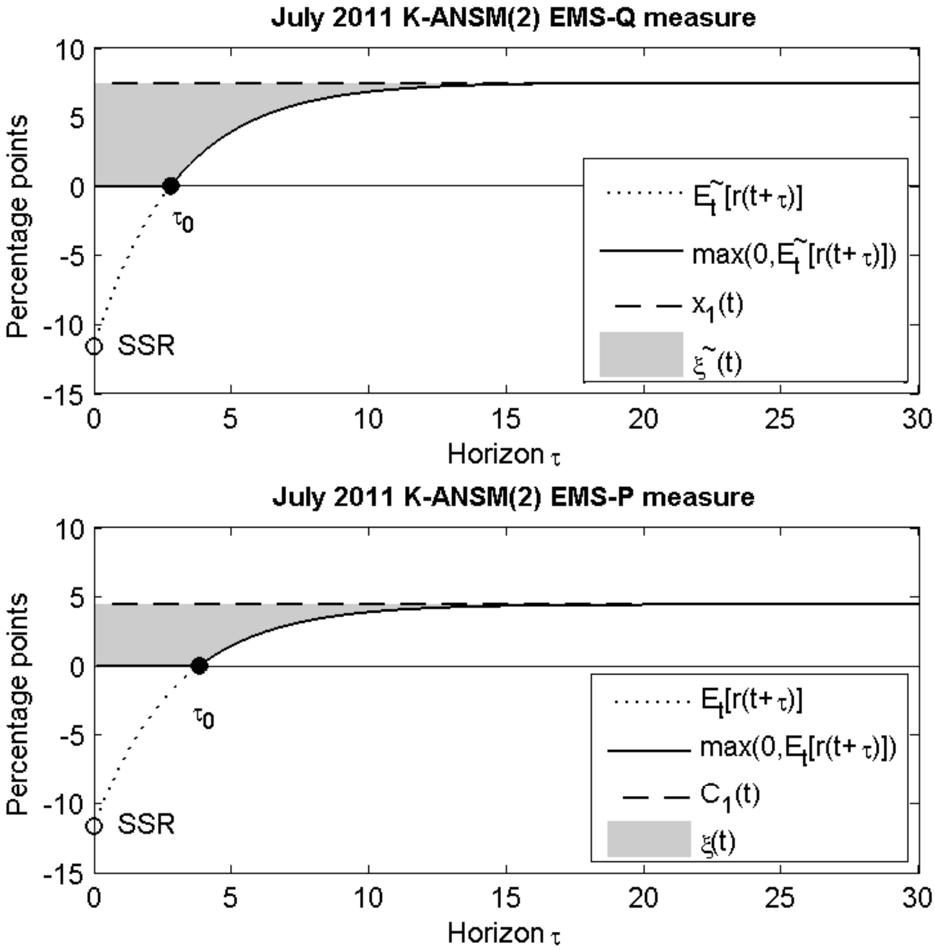

Figure 13: Results for the estimated supplementary K-ANSM(2) under the risk-adjusted $\mathbb{Q}$ measure and the physical $\mathbb{P}$ measure, and the EMS-$\mathbb{Q}$ and EMS-$\mathbb{P}$ measures for July 2011.

When the restrictions $\kappa_1 = 0$ and $\tilde{\kappa}_1 = 0$ are imposed on a shadow/ZLB-GATSM, the difference between the EMS-$\mathbb{Q}$ and EMS-$\mathbb{P}$ measures reflects the effect of the risk premium function. That difference can vary over time, and using that difference may provide a useful quantity in its own right as I discuss in section 5.1. However, note that the specification of $\kappa = \text{diag}[0, \kappa_2]$ and $\tilde{\kappa} = \text{diag}[0, \tilde{\kappa}_2]$ in my illustrative ANSM(2) example implicitly imposes $\Gamma_{11} = 0$, i.e.:

$$\begin{aligned} \tilde{\kappa} &= \kappa + \Gamma \\ \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\kappa}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & \kappa_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_{22} \end{bmatrix} \end{aligned} \tag{39}$$

Therefore the market prices of risk are not allowed to vary with the Level state variable $x_1(t)$, which may be an oversimplified model for practical use.

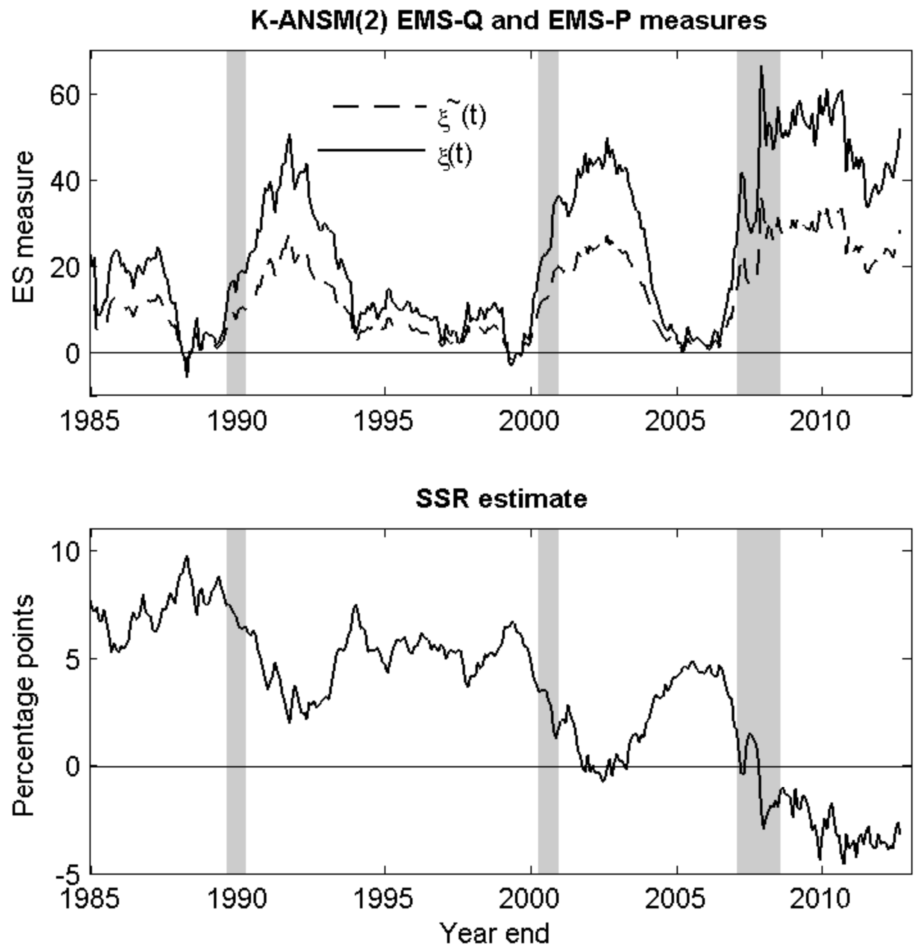

Figure 14: Time series plots of the EMS-$\mathbb{Q}$ and EMS-$\mathbb{P}$ measures and the SSR for the restricted K-ANSM(2) discussed in the text. The restrictions noted in the text result in time series that could be compared to macroeconomic variables.

5 Discussion and future work

In this section, I first summarize the key conclusions from sections 3 and 4 as a precursor to discussing future work that will likely be required to refine and assess EMS measures. I then briefly discuss some alternative EMS measures that could be defined, although none seem to offer the intuition and parsimony of the K-ANSM EMS measures already presented in sections 3 and 4.

5.1 EMS measures already proposed

The key results in section 4 are as follows: (1) the K-GATSM EMS-$\mathbb{Q}$ measures associated with $\tilde{\kappa}_1 = 0$ have stable mathematical properties, an intuitive economic interpretation, and are arguably better measures of the stance of monetary policy in principle than actual short rates, SSRs, single interest rates of other maturities, or interest rate spreads;[16] (2) empirically, the $\tilde{\kappa}_1 = 0$ EMS-$\mathbb{Q}$ measures appear to be more robust than SSRs to different model specifications and choices of data for estimation, although some effects of the

[16] These in-principle points apply equally to B-GATSM EMS measures.

practical choices underlying EMS-$\mathbb{Q}$ measures are still evident; and (3) EMS-$\mathbb{P}$ measures with $\kappa_1 = 0$ have analogous properties to $\tilde{\kappa}_1 = 0$ EMS-$\mathbb{Q}$ measures.

Each of the points mentioned above is subject to further consideration and analysis. For example, one conceptual question that arises from point 3 is which of the $\tilde{\kappa}_1 = 0$ EMS-$\mathbb{Q}$ or $\kappa_1 = 0$ EMS-$\mathbb{P}$ measures is preferable in principle. The EMS-$\mathbb{Q}$ measure is based on a risk-adjusted measure, and so naturally relates to asset prices. However, the EMS-$\mathbb{P}$ measure is based on a physical measure, and so relates to actual quantities faced by economic agents like expected output growth and inflation. Both measures are likely to be useful in different contexts. The implied risk premium measure obtained as the difference between the EMS-$\mathbb{Q}$ and EMS-$\mathbb{P}$ measures may also be useful in its own right, particularly because the effect of unconventional monetary policy is considered to arise from a combination of expected policy rates and risk premiums; e.g. see Woodford (2012) for an overview.

Regarding points 1 and 2, a detailed empirical assessment of EMS measures relative to traditional metrics for monetary policy would be required to determine if EMS measures are potentially useful in the first instance, and then which EMS measure is most useful in practice. One perspective of these assessments would be testing the correlation of EMS measures with the known evolution of monetary policy actions and guidance, over both ZLB and non-ZLB periods. The ultimate test, which cuts to the essence of operating monetary policy with respect to macroeconomic policy targets and/or objectives, would be to assess the inter-relationships of EMS measures with macroeconomic data like inflation, output growth, and exchange rates. The two figures below, from preliminary related work by the author, suggest that the EMS measures perform usefully in practice as monetary policy metrics over both conventional and unconventional monetary policy environments.

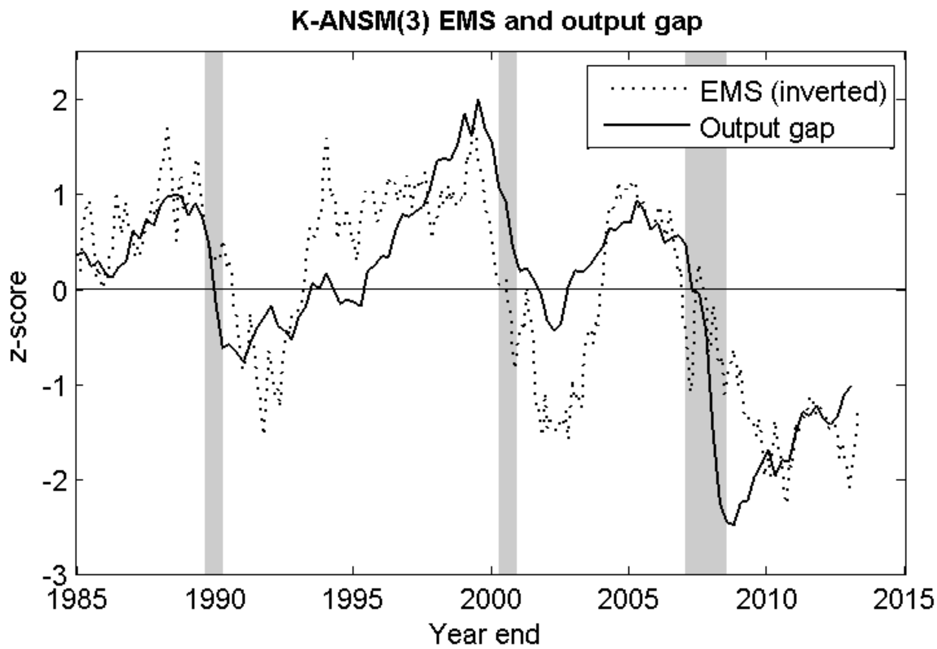

Figure 14b: The U.S. K-ANSM(3) EMS correlates well with the output gap, which is consistent with more stimulatory policy when output is below potential, and more restrictive policy when output is above potential.

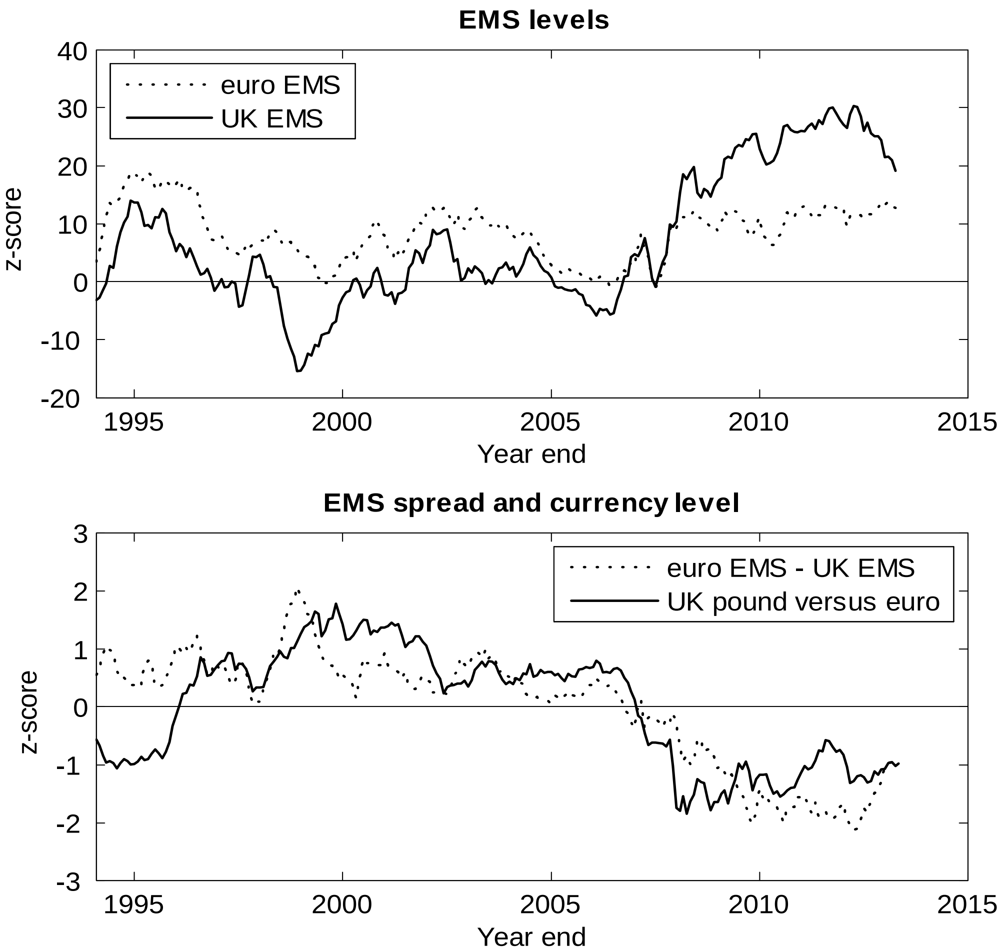

Figure 14c: The U.K. EMS relative to the euro-area EMS correlates well with the U.K. versus the euro currency, which is consistent with relatively more stimulatory policy coinciding with a weaker exchange rate, and relatively more restrictive policy coinciding with a stronger exchange rate.

Also regarding point 2, the sensitivity of EMS measures to the estimated Level state variable raises a potential avenue for improving the EMS measures. Mechanically, if the scope for the cyclical variation of the Level estimate were limited, then the EMS measures would reflect more of the cyclical variation in the yield curve data. Such limits could be obtained using appropriate restrictions and/or using information external to the model and data. For example, survey information on expected market interest rates under the $\mathbb{P}$-measure could be used to improve the estimation of the model, as discussed in Kim and Orphanides (2012). In addition, data related to the macroeconomic quantities that should underlie long-run/neutral interest rates may be exploitable. In particular, surveys of long-term inflation expectations could be incorporated into the $\mathbb{P}$-measure specification of a shadow/ZLB-ANSM.

5.2 Alternative EMS measures

The EMS-$\mathbb{Q}$ measure could be converted into a relative asset price basis as follows:

$$
\begin{aligned}
\tilde{\xi}_{A1}(t) &= \frac{\exp\left(-\int_0^\infty \tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right]\,\mathrm{d}u\right)}{\exp\left(-\int_0^\infty \lim_{\tau\to\infty}\tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right]\,\mathrm{d}u\right)} \\
&= \exp\left(-\int_0^\infty \tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right] - \lim_{\tau\to\infty}\tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right]\,\mathrm{d}u\right) \\
&= \exp\left[-\tilde{\xi}(t)\right]
\end{aligned}
\tag{40}
$$

Figure 15 plots the alternative EMS-$\mathbb{Q}$ measure $\tilde{\xi}_{A1}(t)$ for the K-ANSMs already discussed earlier. Note that the lower values now indicate more stimulus, but otherwise $\tilde{\xi}_{A1}(t)$ captures the same information and has the same underlying foundation as $\tilde{\xi}(t)$.

As noted in section 3.1, I intentionally define the EMS-$\mathbb{Q}$ measure $\tilde{\xi}(t)$ using the expression $\max\left\{0, \tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right]\right\}$ rather than using $\tilde{\mathbb{E}}_t\left[\max\left\{0, \mathrm{r}(t+\tau)\right\}\right]$ which is non-equivalent. An EMS-$\mathbb{Q}$ measure $\tilde{\xi}_{A2}(t)$ defined as:

$$
\tilde{\xi}_{A2}(t) = \int_0^\infty \tilde{\mathbb{E}}_t\left[x_1(t+\tau) - \max\left\{0, x_1(t+\tau) - \mathrm{r}(t+\tau)\right\}\right] \tag{41}
$$

would be unbounded for the case where $\tilde{\kappa}_1 = 0$ and very model sensitive in the case $\tilde{\kappa}_1 \gtrsim 0$. Appendix C.1 contains further details and discussion on these issues, and the principles would apply analogously for the EMS-$\mathbb{P}$ measure $\xi_{A2}(t)$.

An alternative EMS measure $\tilde{\xi}_{A3}(t)$ based on discounting assumed cashflows would at least be mathematically defined in both the $\tilde{\kappa}_1 = 0$ and $\tilde{\kappa}_1 \gtrsim 0$ cases. However, the interest rates used for the discounting would themselves not have any economic interpretation. Appendix C.2 contains further details and discussion on such a measure.

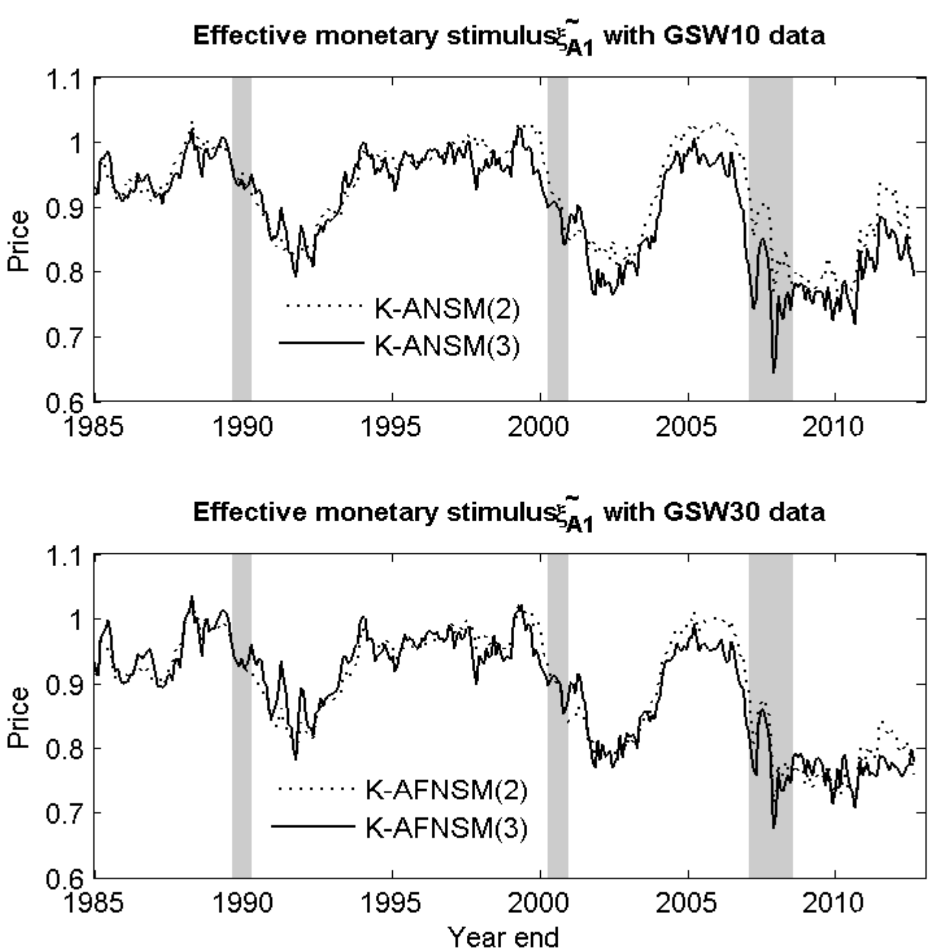

Figure 15: Time series plots of the EMS-$\mathbb{Q}$ measures from figures 4 and 5, but on an asset price basis. The transformation means that lower values imply greater monetary stimulus.

6 Conclusion

In this article, I have introduced the idea of EMS measures based on shadow/ZLB-GATSMs with the restrictions $\tilde{\kappa}_1 = 0$ and $\kappa_1 = 0$ to summarize of the stance of monetary policy. EMS measures aggregate the entire expected path of SSRs truncated at zero relative to their long-run expectation from the model (a proxy for a neutral rate), and are consistent and comparable across conventional/non-ZLB and unconventional/ZLB environments. In principle, EMS measures should be a superior indicator than any particular actual or estimated interest rate, and in practice the EMS measures calculated for two and three factor shadow/ZLB-GATSMs with the restrictions $\tilde{\kappa}_1 = 0$ and $\kappa_1 = 0$ are shown to be more robust than the SSR estimates. Further assessment and potential improvements of EMS measures remains to be undertaken, particularly investigating the inter-relationships of the EMS measure with macroeconomic data.

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A EMS measures for stationary shadow-GATSMs

In this appendix, I discuss EMS measures for stationary shadow/ZLB-GATSMs, and illustrate the principles using the two-factor stationary K-GATSM, or K-GATSM(2), results from Krippner (2013d). In appendix A.1, I show that using a fixed value for the long-run expectation/neutral interest rate leads to EMS measures that lack what I will call "economic meaning". For the purposes of this appendix, a quantity without "economic meaning" should broadly be interpreted as something that doesn't correlate with macroeconomic quantities such as output growth (or related series, like the output gap or recession indicators).[17] However, based on the discussion in Kim and Orphanides (2012)

[17] A more formal definition is beyond the scope of the present article. For that purpose, I am currently developing a macro-finance framework based on a multi-factor version of the Cox, Ingersoll, and Ross (1985a) economy, which is used to justify GATSMs, Cox, Ingersoll, and Ross (1985b)/square-root models, and Gaussian/square-root mixture models.

p .245,[18] if one effectively treats the persistent component of a stationary GATSM as an approximation to the Level component of non-stationary GATSM, then EMS measures can be obtained analogous to the K-ANSMs in sections 3 and 4. I illustrate this in appendix A.2. Note that the exposition and examples I use are for the $\mathbb{Q}$ measure, but the same results hold analogously for the $\mathbb{P}$ measure. Also, the K-GATSM(2) from Krippner (2013d) has a block-diagonal mean-reversion matrix, but I show in appendix B how EMS measures for non-stationary or stationary non-block-diagonal may be calculated.

A.1 Constant long-run SSR expectations

If all eigenvalues of $\tilde{\kappa}$ (and κ) for the shadow-GATSM are greater than zero, then the expected path of the SSR under the risk-adjusted $\mathbb{Q}$ measure is:

$$\tilde{\mathbb{E}}_t \left[\mathrm{r} \left(t + \tau \right) \right] = a_0 + b_0' \exp \left(-\tilde{\kappa} \tau \right) x \left(t \right) \tag{42}$$

and the infinite expectation of equation 42 is:

$$\lim_{\tau \to \infty} \tilde{\mathbb{E}}_t \left[\mathrm{r} \left(t + \tau \right) \right] = a_0 \tag{43}$$

For the K-GATSM(2), using a_0 as the long-run SSR expectation/neutral interest rate gives an EMS-$\mathbb{Q}$ measure of:

$$\tilde{\xi} \left(t \right) = \begin{cases} \int_0^{\tau_0} a_0 \, \mathrm{d}\tau - \int_{\tau_0}^{\infty} x_1 \left(t \right) \cdot \exp \left(-\tilde{\kappa}_1 \tau \right) + x_2 \left(t \right) \cdot \exp \left(-\tilde{\kappa}_2 \tau \right) \, \mathrm{d}\tau & \text{if } \mathrm{r} \left(t \right) < 0 \\ - \int_0^{\infty} x_1 \left(t \right) \cdot \exp \left(-\tilde{\kappa}_1 \tau \right) + x_2 \left(t \right) \cdot \exp \left(-\tilde{\kappa}_2 \tau \right) \, \mathrm{d}\tau & \text{if } \mathrm{r} \left(t \right) \geq 0 \end{cases} \tag{44}$$

Figure 16 illustrates the EMS-$\mathbb{Q}$ measure $\tilde{\xi} \left(t \right)$ as at July 2011 for the K-GATSM(2) estimated with GSW10 and GSW30 data. In both cases, the low rates of mean reversion (i.e. $\tilde{\kappa}_1 = 0.0348$ and $\tilde{\kappa}_1 = 0.0283$ for the GSW10 and GSW30 data respectively) lead to those components relative to the estimate of a_0 dominating the EMS measure. Specifically, the persistent overshoot of $\tilde{\mathbb{E}}_t \left[\mathrm{r} \left(t + \tau \right) \right]$ relative to a_0 in the GSW10 case leads to a large negative EMS-$\mathbb{Q}$ measure of $\tilde{\xi} \left(t \right) = -70.27$, while the persistent undershoot in the GSW30 case leads to a very large positive EMS measure of $\tilde{\xi} \left(t \right) = 462.43$. Those large and persistent deviations of $\mathbb{E}_t \left[\mathrm{r} \left(t + \tau \right) \right]$ relative to a_0 over practical horizons are an initial indication that the K-GATSM(2) EMS measures lack economic meaning.[19]

[18] To quote: "Although this contrasts with models that require the infinite-horizon expectation of the short-term interest rate to vary over time, we note that, as a practical matter, the stationary model we consider is sufficiently flexible that it can accommodate considerable time variation in "long-horizon" forecasts (say 5 to 10 years), and it may be hard to distinguish from nonstationary models even over such long horizons."

[19] The K-ANSM(2) example in section 3.2 has a $\mathbb{P}$ measure K-GATSM(2) with $\kappa_1 = 3.952\text{e-}06$, which leads to a very persistent process for $\mathbb{E}_t \left[\mathrm{r} \left(t + \tau \right) \right]$ and therefore extremely large values of $\xi \left(t \right)$.

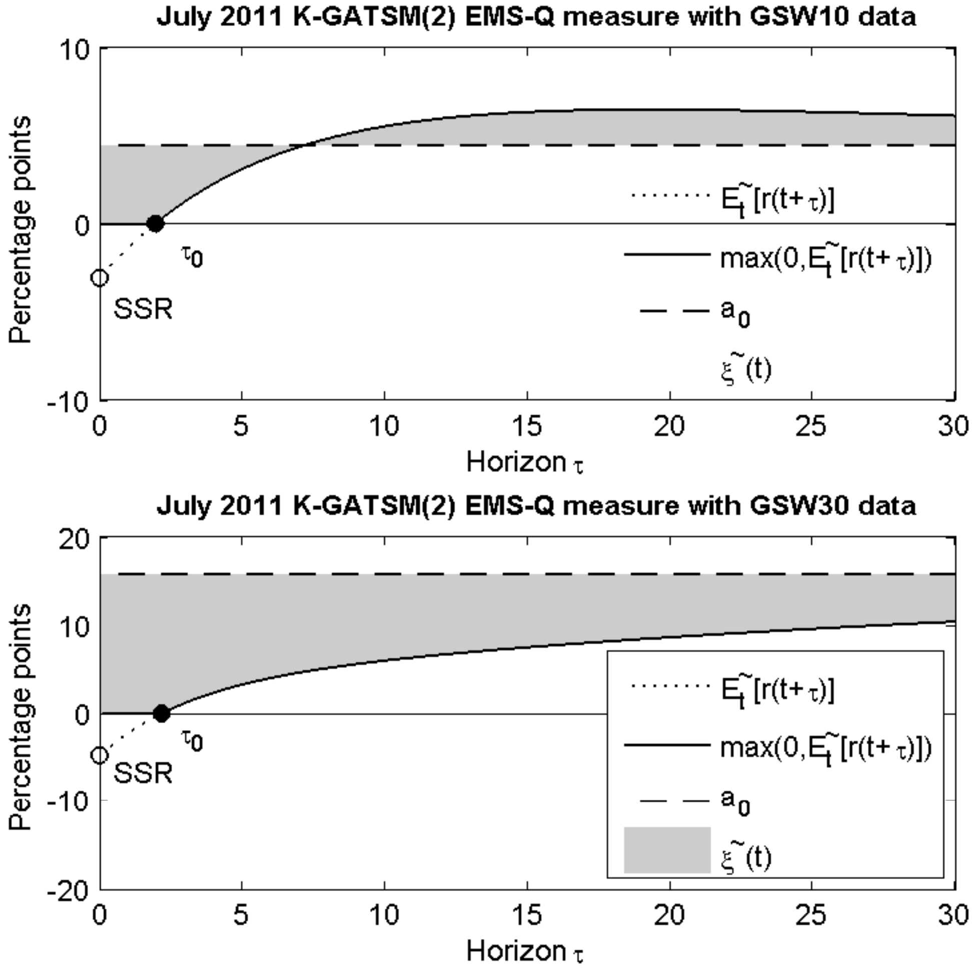

Figure 16: K-GATSM(2) EMS-$\mathbb{Q}$ measures for July 2011 based on a constant long-run SSR expectation, and estimated with GSW10 and GSW30 data. The persistent K-GATSM(2) components associated with $\tilde{\kappa}_1 \gtrsim 0$ relative to the constant dominates the EMS-$\mathbb{Q}$ measures, leading them to lack economic meaning.

A lack of economic meaning is also suggested in figure 17, which illustrates the time series of EMS-$\mathbb{Q}$ measures based on K-GATSM(2) estimates using GSW10 and GSW30 data. The EMS-$\mathbb{Q}$ measures are standardized as z scores (i.e. $\left\{\tilde{\xi}(t) - \text{mean}\left[\tilde{\xi}(t)\right]\right\} / \text{stdev}\left[\tilde{\xi}(t)\right]$) so they can be plotted on the same scale. Both series show an upward trend over time which, if taken literally, would imply a steady easing of actual and/or anticipated monetary conditions over the sample period, and which is inconsistent with the NBER recession indicators. Of course, that upward trend actually reflects the general downward trend in the yield curve data relative to the estimated constant a_0, which is used as the long-run expectation of $\mathbb{E}_t[\mathrm{r}(t+\tau)]$. The time trend in the EMS measures will occur irrespective of the estimate of a_0 (e.g. one could argue that $a_0 = 15.71\%$ for the GSW30 data seems high); the only requirement is that a_0 is constant, which is the case for K-GATSMs.

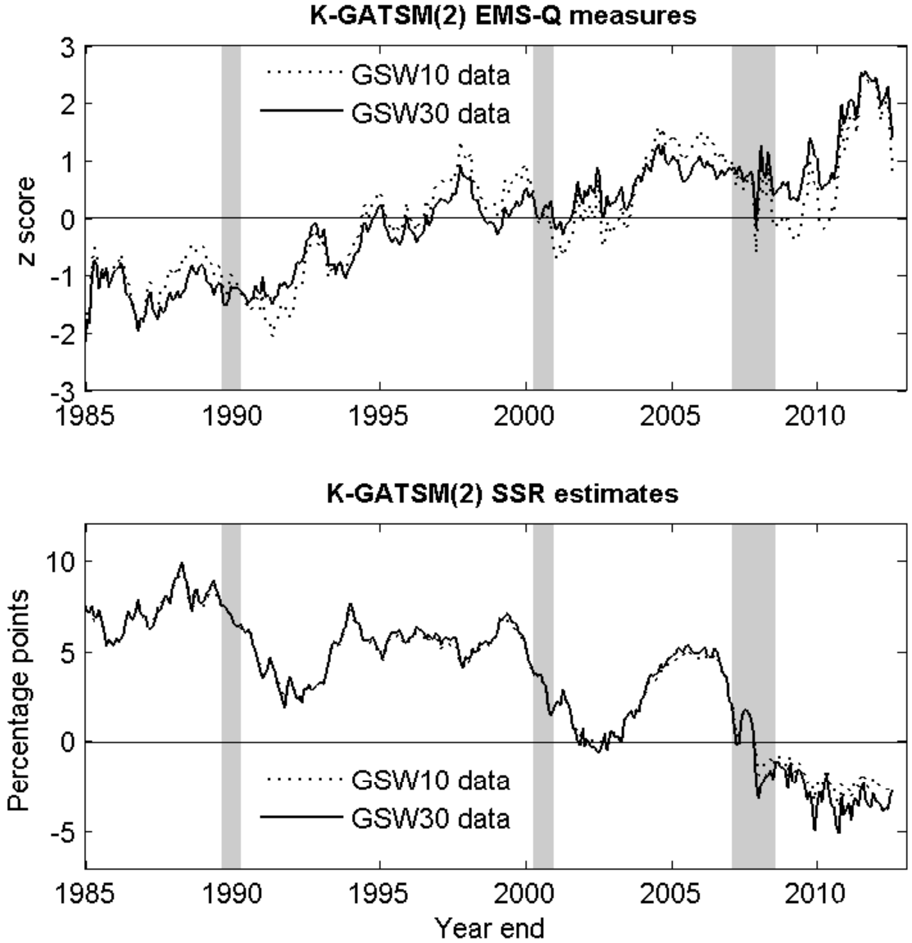

Figure 17: Time series plots of K-GATMS(2) SSRs and EMS-$\mathbb{Q}$ measures based on a constant long-run expectation, and estimated with GSW10 and GSW30 data. The time trend in the EMS-$\mathbb{Q}$ measures suggest that they lack economic meaning.

In general, as previously mentioned in section 3.1, shadow-GATSMs and GATSMs inevitably turn out to have $\tilde{\kappa}_1 \gtrsim 0$. Therefore, the issues already illustrated above for the K-GATSM(2) with a constant long-run SSR expectation will hold generally for shadow/ZLB-GATSMs when calculating EMS-$\mathbb{Q}$ measures. One potential resolution to the lack of economic meaning would be to allow a time-varying estimate of a_0, perhaps by incorporating a regime switching model. That option would be more appealing from an economic perspective, but it would create a less parsimonious model from a practical perspective. In any case, figure 11 in section 3.5 indicates that any time variation in a_0 would have to replicate a highly persistent process, so simply imposing the restriction $\tilde{\kappa}_1 = 0$ with a K-ANSM would present a more pragmatic solution.

A.2 Time-varying long-run SSR expectations

If a low mean-reversion process to $x_1(t)$ is used to approximate a time-varying long-run SSR expectation/neutral interest rate, then $\lim_{\tau\to\infty} \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)]$ becomes:

$$\lim_{\tau\to\infty} \tilde{\mathbb{E}}_t[\mathrm{r}(t+\tau)] = a_0 + x_1(t)\exp(-\kappa_1\tau) \tag{45}$$

and the EMS-$\mathbb{Q}$ measure therefore becomes the analogue of equation 22, i.e.:[20]

$$\begin{aligned}\tilde{\xi}(t) &= \int_0^\infty \left(a_0 + x_1(t)\exp(-\kappa_1\tau) - \max\left\{0, \tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right]\right\}\right) \mathrm{d}\tau \\ &= \int_0^\infty \max\left\{a_0 + x_1(t)\exp(-\kappa_1\tau), b'_{0,L}\exp(-\tilde{\kappa}_L\tau)\, x_L(t)\right\} \mathrm{d}\tau \end{aligned} \quad (46)$$

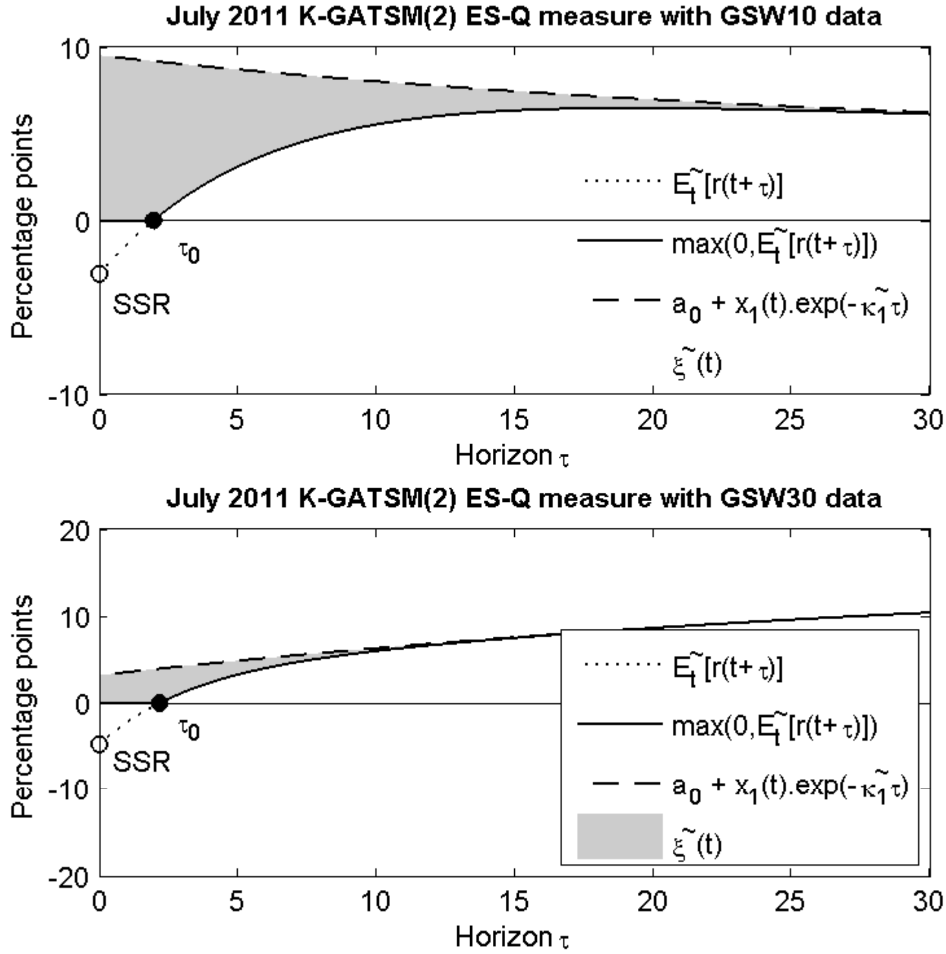

Figure 18: K-GATSM(2) EMS-$\mathbb{Q}$ measures for July 2011 based on approximate time-varying long-run expectations, and estimated with GSW10 and GSW30 data. The very large values from figure 13 are now absent.

Equation 46 is well-defined mathematically for any horizon, as figure 18 illustrates as at July 2011 for the K-GATSM(2) estimated with GSW10 and GSW30 data. Nevertheless, there are two apparent downsides relative to the K-ANSM(2). First, $a_0 + x_1(t)\exp(-\kappa_1\tau)$ can really only be viewed as a long-run SSR expectation/neutral interest rate for relatively short horizons, over which the expression will remain approximately constant. In figure 18, $x_1(t)\exp(-\kappa_1\tau)$ has some attenuation even out to 10 years, and it becomes much more noticeable out to 30 years. Second, the EMS-$\mathbb{Q}$ measures are not as robust in outright terms between the estimations with different data. Therefore, in figure 19, I have again

[20] Equations 42, 43, and 46 will still apply if $\tilde{\kappa}$ contains repeated eigenvalues, e.g. $\tilde{\kappa}_2 = \tilde{\kappa}_3$ for the three-factor shadow-GATSM in Wu and Xia (2013). In that case, the corresponding components of $\tilde{\mathbb{E}}_t\left[\mathrm{r}(t+\tau)\right]$ take the form detailed in section 3.3.

standardized as z scores the time series of EMS-$\mathbb{Q}$ measures estimated with GSW10 and GSW30 data. The two series, albeit standardized, track each other very closely. More importantly, EMS-$\mathbb{Q}$ measures also line up with the economic recessions, as discussed in section 3.4, indicating that they are correlated with output growth and/or output gap data.

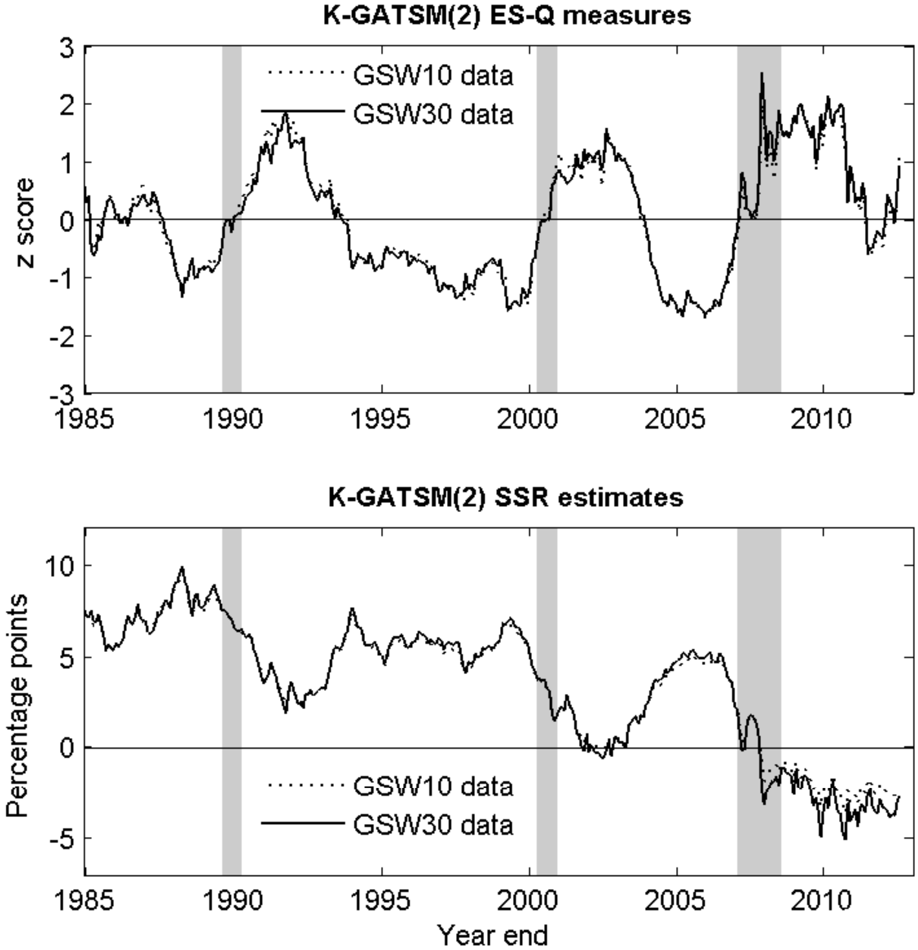

Figure 19: Time series plots of K-GATMS(2) SSRs and EMS-$\mathbb{Q}$ measures based on approximate time-varying long-run expectations, and estimated with GSW10 and GSW30 data. The time trend evident in figure 14 is now absent.

In summary, stationary K-GATSMs (or B-GATSMs) could be used to obtain EMS measures with economic meaning. However, EMS measures based on models with an imposed eigenvalue of zero appear to offer a more pragmatic solution; in practice they produce EMS measures that are robust between different models and data, and the concept of the time-varying long-run SSR expectation/neutral interest rate from the framework also holds over any horizon.

B EMS measures from non-block-diagonal specifications

I have used block-diagonal specifications of the mean-reversion matrices in this article to simplify the notation and resulting expressions. However, EMS measures can be calculated analogously with non-block-diagonal specifications. I illustrate this for $\mathbb{P}$ measures, because estimation restrictions for GATSMs and shadow-GATSMs are typically applied under the $\mathbb{Q}$ measure, as mentioned at the end of section 2.2, leaving more flexibility under the $\mathbb{P}$ measure. However, the results would apply analogously for non-block-diagonal specifications of mean-reversion matrices under the $\mathbb{Q}$ measure.

For any K-GATSM (or B-GATSM), the expected path of the state variables under the physical $\mathbb{P}$ measure is:

$$\begin{aligned}\mathbb{E}_t\left[x\left(t+\tau\right)\right] &= \theta+\exp\left(-\kappa\tau\right)\left[x\left(t\right)-\theta\right]\\ &= \theta+\exp\left(-VDV^{-1}\tau\right)\left[x\left(t\right)-\theta\right]\\ &= \theta+V\exp\left(-D\tau\right)V^{-1}\left[x\left(t\right)-\theta\right]\end{aligned}\tag{47}$$

where $VDV^{-1}=\kappa$ is a Jordan decomposition (which allows for repeated eigenvalues), with an $N\times N$ matrix V containing the eigenvectors in columns and an $N\times N$ matrix D containing the blocks of Jordan matrices. Note that the eigensystem decomposition could potentially result in pairs of complex conjugates (if no additional restrictions are applied), and that the real parts of the eigenvalues, i.e. $\text{real}(\kappa_i)>0$, need to be positive to ensure a mean-reverting process. In practice, as with all of the examples in the present article, κ_1 inevitably has a single real entry, and so I assume that in what follows. Hence, I denote D as:

$$D=\begin{bmatrix}\kappa_1 & 0\\ 0 & \kappa_L\end{bmatrix}\tag{48}$$

where κ_1 is a scalar, and κ_L is the $(N-1)\times(N-1)$ lower block of D. Hence:

$$\exp\left(-D\tau\right)=\begin{bmatrix}\exp\left(-\kappa_1\tau\right) & 0\\ 0 & \exp\left(-\kappa_L\tau\right)\end{bmatrix}\tag{49}$$

Equation 47 may be re-expressed with a block diagonal mean-version matrix by premultiplying equation 47 by V^{-1}, i.e.:

$$\begin{aligned}V^{-1}\mathbb{E}_t\left[x\left(t+\tau\right)\right] &= V^{-1}\theta+\exp\left(-\kappa\tau\right)V^{-1}\left[x\left(t\right)-\theta\right]\\ \mathbb{E}_t\left[x^*\left(t+\tau\right)\right] &= \theta^*+\exp\left(-\kappa\tau\right)\left[x^*\left(t\right)-\theta^*\right]\end{aligned}\tag{50}$$

where $\mathbb{E}_t\left[x^*\left(t+\tau\right)\right]=V^{-1}\mathbb{E}_t\left[x\left(t+\tau\right)\right]$, $x^*\left(t\right)=V^{-1}x\left(t\right)$, and $\theta^*=V^{-1}\theta$. Therefore the expected path of the state variables is equivalently represented by the process:

$$\begin{bmatrix}\mathbb{E}_t\left[x_1^*\left(t+\tau\right)\right]\\ \mathbb{E}_t\left[x_L^*\left(t+\tau\right)\right]\end{bmatrix}=\begin{bmatrix}\theta_1^*\\ \theta_L^*\end{bmatrix}+\begin{bmatrix}\exp\left(-\kappa_1\tau\right) & 0\\ 0 & \exp\left(-\kappa_L\tau\right)\end{bmatrix}\left(\begin{bmatrix}x_1^*\left(t\right)\\ x_L^*\left(t\right)\end{bmatrix}-\begin{bmatrix}\theta_1^*\\ \theta_L^*\end{bmatrix}\right)\tag{51}$$

where the top line contains the expectations process for $x_1^*\left(t\right)$, and the bottom line contains the expectations process for the remaining elements of $x^*\left(t\right)$, which I denote as the $(N-1)\times 1$ vector $x_L^*\left(t\right)$.

If $\kappa_1=0$, then $\exp\left(-\kappa_1\tau\right)$ in equation 49 becomes 1, and the GATSM is non-stationary with a random walk process for $x_1^*\left(t\right)$, i.e.:

$$\mathbb{E}_t\left[x_1^*\left(t+\tau\right)\right]=\theta_1^*+\left[x_1^*\left(t\right)-\theta_1^*\right]=x_1^*\left(t\right)\tag{52}$$

Otherwise, if $\kappa_1\gtrsim 0$, $\mathbb{E}_t\left[x_1^*\left(t+\tau\right)\right]$, the GATSM is stationary with a persistent mean-reverting process for $x_1^*\left(t\right)$, i.e.:

$$\mathbb{E}_t\left[x_1^*\left(t+\tau\right)\right]=\theta_1^*+\exp\left(-\kappa_1\tau\right)\left[x_1^*\left(t\right)-\theta_1^*\right]\tag{53}$$

The vector $x_L^*(t)$ follows a mean-reverting process, i.e.:

$$\mathbb{E}_t\left[x_L^*(t+\tau)\right] = \theta_L^* + \exp\left(-\kappa_L\tau\right)\left[x_L^*(t) - \theta_L^*\right] \tag{54}$$

Note that these expectations could also be computed directly using matrix exponentials, which is a standard function available in MatLab. However, using a Jordan decomposition is faster because it allows for vectorized evaluations of all expectation horizons simultaneously using scalar exponentials.

The expected path of the shadow short rate $\mathbb{E}_t\left[\mathrm{r}(t+\tau)\right]$ may be expressed in terms of $\mathbb{E}_t\left[x^*(t+\tau)\right]$ as follows:

$$\begin{aligned}\mathbb{E}_t\left[\mathrm{r}(t+\tau)\right] &= a_0 + b_0'\mathbb{E}_t\left[x(t+\tau)\right] \\ &= a_0 + b_0'VV^{-1}\mathbb{E}_t\left[x(t+\tau)\right] \\ &= a_0 + b_0^{*\prime}\mathbb{E}_t\left[x^*(t+\tau)\right] \\ &= a_0 + b_{0,1}^*\mathbb{E}_t\left[x_1^*(t+\tau)\right] + b_{0,L}^{*\prime}\mathbb{E}_t\left[x_L^*(t+\tau)\right] \\ &= a_0 + b_{0,1}^*\mathbb{E}_t\left[x_1^*(t+\tau)\right] \\ &\quad + b_{0,L}^{*\prime}\left\{\theta_L^* + \exp\left(-\kappa_L\tau\right)\left[x_L^*(t) - \theta_L^*\right]\right\}\end{aligned} \tag{55}$$

where I have left $\mathbb{E}_t\left[x_1^*(t+\tau)\right]$ generic to allow for either non-stationary or stationary specifications, and $b_0^{*\prime} = b_0'V$. b_0^* is an $N\times 1$ vector composed of the first element $b_{0,1}^*$ and the $(N-1)\times 1$ lower-block vector $b_{0,L}^*$. Note that if $\kappa_1 > 0$ and $\tilde{\kappa}_1 = 0$, which is the case for K-ANSMs with the restriction $\mathrm{real}(\kappa_i) > 0$ that is often applied in practice,[21] then the restriction $a_0 = 0$ applies. The restriction $a_0 = 0$ will also apply if $\kappa_1 = 0$.

With $\kappa_1 = 0$, the long-run expectation of equation 55 is:

$$\begin{aligned}\lim_{\tau\to\infty}\mathbb{E}_t\left[\mathrm{r}(t+\tau)\right] &= b_{0,1}^*\lim_{\tau\to\infty}\mathbb{E}_t\left[x_1^*(t+\tau)\right] + b_{0,L}^{*\prime}\lim_{\tau\to\infty}\mathbb{E}_t\left[x_L^*(t+\tau)\right] \\ &= b_{0,1}^{*\prime}x_1^*(t) + b_{0,L}^{*\prime}\theta_L^*\end{aligned} \tag{56}$$

and therefore the EMS-$\mathbb{P}$ measure is:

$$\begin{aligned}\xi(t) &= \int_0^\infty \left(b_{0,1}^*x_1^*(t) + b_{0,L}^{*\prime}\theta_L^* - \max\left\{0, \mathbb{E}_t\left[\mathrm{r}(t+\tau)\right]\right\}\right)\mathrm{d}\tau \\ &= \int_0^\infty \max\left\{b_{0,1}^*x_1^*(t) + b_{0,L}^{*\prime}\theta_L^*, -b_{0,L}^{*\prime}\exp\left(-\kappa_L\tau\right)\left[x_L^*(t) - \theta_L^*\right]\right\}\,\mathrm{d}\tau\end{aligned} \tag{57}$$

If $\kappa_1 \gtrsim 0$, the GATSM is stationary and the approximate long-run expectation of 55 is:

$$\begin{aligned}\lim_{\tau\to\infty}\mathbb{E}_t\left[\mathrm{r}(t+\tau)\right] &= a_0 + b_{0,1}^*\mathbb{E}_t\left[x_1^*(t+\tau)\right] + b_{0,L}^{*\prime}\mathbb{E}_t\left[x_L^*(t+\tau)\right] \\ \mathrm{r}_\infty &= a_0 + \theta_L^* + \exp\left(-\kappa_L\tau\right)\left[x_L^*(t) - \theta_L^*\right] + b_{0,L}^{*\prime}\theta_L^*\end{aligned} \tag{58}$$

where r_∞ is introduced for notational convenience. The EMS-$\mathbb{P}$ measure is therefore:

$$\begin{aligned}\xi(t) &= \int_0^\infty \left(\mathrm{r}_\infty - \max\left\{0, \mathbb{E}_t\left[\mathrm{r}(t+\tau)\right]\right\}\right)\mathrm{d}\tau \\ &= \int_0^\infty \max\left\{\mathrm{r}_\infty, -b_{0,L}^{*\prime}\exp\left(-\kappa_L\tau\right)\left[x_L^*(t) - \theta_L^*\right]\right\}\,\mathrm{d}\tau\end{aligned} \tag{59}$$

In summary, EMS measures from K-GATSMs specified with non-block-diagonal mean-reversion matrices are analogous to those with block-diagonal specifications, but matrix exponentials or Jordan decompositions are required for the calculation process.

[21] See, e.g Christensen, Diebold, and Rudebusch (2011) footnote 15 or Krippner (2013d).

C Alternative EMS measures

This appendix provides further discussion on the alternative EMS measures mentioned at the end of section 5.2. I have used the $\mathbb{Q}$ measure for the discussions, but the comments would apply analogously under the $\mathbb{P}$ measure.

C.1 Alternative 2

The expression $\tilde{\mathbb{E}}_t[\max\{0, \mathrm{r}(t+\tau)\}]$ does not have a long-run expectation when $\tilde{\kappa}_1 = 0$, because $x_1(t+\tau)$ has a normal distribution with a mean $x_1(t)$ and a standard deviation that grows as $\sigma_1\sqrt{\tau}$. Therefore:

$$\lim_{\tau\to\infty} \tilde{\mathbb{E}}_t[\max\{0, \mathrm{r}(t+\tau)\}] = \lim_{\tau\to\infty} \tilde{\mathbb{E}}_t[\max\{0, x_1(t+\tau)\}] \tag{60}$$

would be an unbounded positive quantity. The same issue arises for an alternative EMS measure $\tilde{\xi}_{A2}(t)$ based directly on $\tilde{\mathbb{E}}_t[x_1(t) - \max\{0, \mathrm{r}(t+\tau)\}]$. For example, using the ANSM(2) to illustrate, one would obtain the following expression:

$$\begin{aligned} \tilde{\xi}_{A2}(t) &= \int_0^\infty \tilde{\mathbb{E}}_t[x_1(t+\tau) - \max\{0, x_1(t+\tau) - \mathrm{r}(t+\tau)\}] &(61)\\ &= \int_0^\infty \tilde{\mathbb{E}}_t[\max\{x_1(t+\tau), -x_2(t+\tau)\}] &(62) \end{aligned}$$

$\tilde{\xi}_{A2}(t)$ will be an unbounded positive quantity because $x_1(t+\tau)$ is unbounded on the positive domain, and that property translates into $\max\{x_1(t+\tau), -x_2(t+\tau)\}$. At the same time, $\max\{x_1(t+\tau), -x_2(t+\tau)\}$ will be bounded below by $-x_2(t+\tau)$, which converges to a finite distribution because $x_2(t+\tau)$ follows a mean-reverting process. In addition, $x_1(t+\tau)$ can adopt negative values, which would be questionable from an economic perspective.

If all $\tilde{\kappa}_i > 0$, then an EMS measure based on $\tilde{\mathbb{E}}_t[\max\{0, \mathrm{r}(t+\tau)\}]$ would be mathematically defined. However, the state variable $x_1(t+\tau)$ associated with the eigenvalue $\tilde{\kappa}_1 \gtrsim 0$ would have the following mean and standard deviation:

$$\text{mean}[x_1(t+\tau)] = \exp(-\tilde{\kappa}_1\tau) \cdot x_1(t) \tag{63}$$

$$\text{stdev}[x_1(t+\tau)] = \sigma_1\sqrt{\frac{1}{2\tilde{\kappa}_1}[1 - \exp(-2\tilde{\kappa}_1\tau)]} \tag{64}$$

The value of stdev$[x_1(t+\tau)]$ would therefore still be large relative to mean$[x_1(t+\tau)]$ for longer horizons, resulting in the distribution on the positive domain dominating the calculation of $\tilde{\xi}_{A2}(t)$. In other words, $\tilde{\xi}_{A2}(t)$ would take on large values, sensitive to the precise magnitude of small estimates of $\tilde{\kappa}_1$. Therefore, EMS-$\mathbb{Q}$ measures based on $\tilde{\mathbb{E}}_t[\max\{0, \mathrm{r}(t+\tau)\}]$ with either $\tilde{\kappa}_1 = 0$ or $\tilde{\kappa}_1 \gtrsim 0$ would lack an economic interpretation.

C.2 Alternative 3

A potential alternative to the EMS measure proposed in appendix B.1 that would work mathematically in an cases would be to define a measure $\tilde{\xi}_{A3}(t)$ as the analogue of an asset price by discounting an assumed stream of cashflows with the ZLB term structure. For example, assuming a unit cashflow in continuous time would give the following expression:

$$\tilde{\xi}_{A3}(t) = \int_0^\infty \exp\left[-\underline{\mathrm{R}}(t,u)\, u\right] \mathrm{d}u \tag{65}$$

where $\underline{\mathrm{R}}(t,u)$ is given in equation 14. Because $\underline{\mathrm{R}}(t,u) > 0$, $\tilde{\xi}_{A3}(t)$ is a bounded quantity even if $\underline{\mathrm{R}}(t,u)$ is unbounded (which is the case for B-GATSMs with $\tilde{\kappa}_1 = 0$; K-GATSMs interest rates revert to zero for very long horizons). However, such an EMS measure would need to justify any particular assumed stream of cashflows, and there would be no concept of a neutral interest rate underlying it.