

Measuring Provisions for Collateralised Retail Lending

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Abstract

This paper develops a simple model based on an options approach to measure provisions covering expected losses of collateralised retail lending due to default. The dynamics of the probability of default of retail loans is allowed to follow a mean-reverting random process, which captures the characteristics of an economic cycle. Based on the data of the residential mortgage market in Hong Kong, the proposed dynamics of the probability of default is consistent with the empirical findings. A closed-form formula of the model is derived and used to calculate the required provision for a pool of retail loans with the same type of collateral. The numerical results show that the loan-to-value ratio, correlation between the collateral value and the probability of default of borrowers in the pool, volatility of the collateral value, mean-reverting process of the probability of default and time horizon are the important factors for measuring provisions. As the parameters associated with these factors are in general available in banks' databases of their retail loan portfolios, the model could be a useful quantitative tool for measuring provisions.

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1. Introduction

In many countries, the rules for loan provisioning do not aim to capture losses accrued in banks at an early stage, but rather to consider “objective” factors that could be taken into account by their bank supervisors. Some countries provide principle-based rules, with only general guidance on how to determine adequate provisioning. This approach is common in the European Union. In contrast, countries that issue detailed regulations on loan classification often define quantitative minimum provisioning requirements. Most emerging markets take this approach (see World Bank, 2002). The rationale behind issuing detailed regulatory parameters could be to level the playing field or make bank regulations more easily enforceable. Under the second approach, collateral is taken into account when classifying a loan, for example, to a more favourable category than that reflecting its own risk and determining the level of provisions accordingly. The regulator in Australia on the other hand allows banks to set provisions based on their internal models.

While a central feature of provisioning systems is typically to refer to losses that have already been incurred or are anticipated with a high degree of confidence, provisioning requirements may differ significantly for several factors. One of the factors is whether provisioning requirements aim at addressing only losses that follow from visible and identifiable events, or at establishing provisions for expected losses. Another factor is if and how banks are expected to factor in the value of collateral. In many countries, the value of collateral is then subtracted from the required provisions to determine the level of the actual provisions to be established.

Specific provisioning requirements are often designed for certain portfolio segments, such as retail loans including residential mortgage loans and credit card lending. Several countries (for example Australia, France, Korea, the Netherlands, Saudi Arabia and Singapore) do not require retail loans to be classified and

provisioned on an individual basis but allow them to be assessed on a pooled basis. In Australia, for example, management is allowed to deal with small consumer loans on a portfolio basis.

The Basel Committee on Banking Supervision is responsible for proposing regulatory requirements, including capital and provisioning requirements, for internationally active banks. Typically, bank supervisors around the world adopt the guidelines put forth by the Basel Committee. The Basel Committee first proposed the Basel New Capital Framework, also known as Basel II, in June 1999, with revisions in January 2001 and June 2004 (Basel, 2004). By year-end 2006, Basel II is expected to replace the current Basel Accord. Both the current and new capital adequacy frameworks are based on the concept of a capital ratio where the numerator represents the amount of capital of the bank has available and the denominator is a measure of the risks faced by the bank and is referred to as risk-weighted assets. The resulting capital ratio must be no less than 8%.

According to the proposals in Basel II, banks will be allowed to calculate regulatory capital charges for their credit exposures, including those in their retail portfolios, using the standardised approach or the internal ratings-based (IRB) approach. The standardised approach allows less sophisticated banks to use external credit ratings to classify their corporate, bank and sovereign assets into risk classes and to apply different defined risk weights to other assets including retail exposures. Over time, banks are expected to evolve to the IRB approach, which rely on the bank's own experience in determining the risk components of various asset classes.

The IRB calculation of risk-weighted assets for credit exposures relies on four basic risk components: (i) probability of default (PD), which measures the likelihood that the borrower will default over a given time horizon; (ii) loss-given-default (LGD), which measures the proportion of the exposure (after taking the presence and type of

collateral into account) that will be lost if a default occurs; (iii) exposure-at-default, which measures the bank's exposure at the time of default; and (iv) effective maturity. Different risk-weight functions based on the risk components are used by the IRB bank for calculating the corresponding risk-weighted assets of different types of credit exposures. There are three distinct IRB risk-weight functions for different classes of retail exposures: (i) residential mortgages; (ii) revolving credit, and (iii) other retail loans.

Under the framework of Basel II, the IRB banks will also be required to compare their actual provisions with expected losses (see Basel, 2004). Any shortfall (i.e. the expected loss amount exceeds the provision amount) should be deducted from Tier 1 and Tier 2 capital of the bank and any excess (i.e. the provision amount exceeds the expected loss amount) will be eligible for inclusion in Tier 2 capital subject to a cap set by individual bank supervisors. It is therefore important to ensure adequate provisions being provided by banks against expected losses. Basel II defines expected loss as 12.5 times PD times LGD times exposure-at-default (see Basel, 2004). This makes the assumption that the PD and LGD are independent variables, i.e. uncorrelated. The time horizon of the PD and LGD estimates is defined to be one year. It is however noted that defaults are likely to be clustered during times of economic distress and LGD may be correlated with default rates. For example, an increase in defaults in residential mortgage loans leads to an increase in the supply of properties associated with those defaulted loans, and correspondingly to a reduction in their prices and to larger losses for banks.

The effects of the correlation between PD and LGD (including collateral value) on credit risk measures have been considered in the context of corporate loans in recent years. In Frye's (2000a) structural model which draw from the conditional approach suggested by Finger (1999) and Gordy (2000), collateral and asset values of

firms (i.e. borrowers) are modelled using a single index based on a systematic and an idiosyncratic risk factor (the state of the economy). Correlations between PD and LGD result from joint dependence of collateral and asset value of borrowers on systematic risk factors arising from economic cycles. Frye's (2000b) empirical analysis shows a strong positive correlation between default rates and LGD for corporate bonds. The results allow him to conclude that the economic cycle can produce a double misfortune involving greater-than-average default rate and poor-than-average recoveries.

By incorporating collateral value uncertainty to LGD, Jokivuolle and Peura (2003) propose a model of risky debt in which collateral value is correlated with the PD of a borrower. The borrower's PD is based on the default mechanism proposed by Merton (1974) where the borrower defaults its debt if its asset value is less than its outstanding liability at the maturity of the debt. Their numerical studies demonstrate the importance of factors such as collateral value volatility and correlation between collateral value and the borrower's asset value for the estimation of credit risk quantity. On a portfolio basis, Altman *et al.* (2002) use a US corporate bond database covering the period 1982-2000 and find strong evidence of a positive relationship between PD and LGD. They explore through simulation analysis what effect incorporating the positive correlation between PD and LGD has on the value-at-risk for a broadly representative commercial loan portfolio. For their particular simulations they find that setting the correlation between the PD and LGD to zero (as is usual practice), rather than to its estimated value, leads to a reduction in the value-at-risk of at least one quarter.

Although several models and studies exist to guide the effects of the relationship between PD and LGD on credit losses of corporate loans, the body of research on retail credit risk measurement is quite sparse according to the studies

conducted by Allen *et al.* (2004). In view of this observation, Allen *et al.* suggest that techniques of credit risk measurement (such as KMV's Portfolio Manager¹ and Credit Suisse Financial Products' Credit Risk Plus²) for corporate loans could be applied to retail loans. As PD is also assumed to be independent from LGD in these two techniques, the corresponding expected losses remain simply measured as PD times LGD.

Apart from correlation between PD and LGD, the time horizon of the expected-loss measure, which is defined in Basel II as one year, also raises a concern with underestimating provisioning requirements. In order to ensuring adequate level of provisions being provided by banks for expected losses, it is necessary for banks to measure expected losses of retail loans for different time horizons, particularly for long-term secured lending such as residential mortgage loans.

While losses on any single retail loan will not cause a bank to become insolvent, Gross and Souleles (2002) find that retail borrowers were increasing willing to default on their debt, in large part because of the falling social and legal costs of default. This implies that the provisioning requirements of expected losses for individual banks which are active in retail lending could increased due to higher default rates of the banks' retail loans. The Basel Committee (2001) realises that the longer-term viability of the IRB framework would be enhanced by further international agreement on standards for provisioning of expected loss, as capital adequacy critically depends on accurate valuation of banks' assets and liabilities.

¹ A comparative analysis of these techniques can be found in Crouhy *et al.* (2000). KMV's Portfolio Manager is based on the work of Black and Scholes (1973) and Merton (1974) in the pricing of corporate bonds using a contingent-claim framework. In Black-Scholes-Merton's structural framework, a firm's market value of total assets is observable in principle. Furthermore default happens if the total asset value is lower than the value of liabilities. Default risk is therefore equivalent to a European put option on the firm's asset value.

² Credit Risk Plus is on the other hand based on the theoretical underpinnings of reduced-form models in which default time is a stopping time of some given hazard rate process and the payoff upon default is specified exogenously. These models have been considered by Jarrow and Turnbull (1995), Jarrow *et al.* (1997), and Duffie and Singleton (1997).

This would make it easier to revisit the IRB framework should future efforts to consider changes in the definition of regulatory capital and/or more harmonised provisioning rules be undertaken. In view of the above developments, the measurement of provisions against expected losses of retail lending secured by collateral would be important for improving the capital adequacy framework for banks.

The purpose of this paper is to develop a model for measuring provisions of a pool of collateralised retail loans which have the same collateral type (e.g. residential properties) and broadly the same loan-to-value ratio. The model follows the contingent-claim approach of pricing options developed by Black and Scholes (1973) and Merton (1973). The pool of loans is equivalent to a put option written by a bank to its borrowers, where the borrowers could walk away by letting the bank reprocess the collateral upon default. The strike of the put option is the outstanding amount of the loans. When the borrowers default their loans, the loss incurred in the bank is the amount of the loans less the value recovered from the sales of the collateral securing the loans. The loss is the same as a standard payoff of a put option. The provision is therefore equivalent to the option premium multiplied by the PD of borrowers in the pool. The model thus takes account of the stochasticity of the collateral value.

The PD of borrowers in a pool over a given time horizon is another stochastic variable in the model. This means that there is a probability that each loan in the pool will default within a time period. The dynamics of the PD of retail loans could be assumed to follow a mean-reverting random process, which captures the characteristics of an economic cycle. Campbell and Dietrich (1983) find that during the 1960s and 1970s default for insured residential mortgages has been significantly influenced by changes in regional rates of unemployment in the US. This finding is consistent with the study by Agarwal and Liu (2000), that unemployment is a

significant determinant of a household's delinquency and bankruptcy decision by focusing on the credit card market in the US. As changes in unemployment rates reflect the states of an economic cycle, the findings provide empirical evidence that the PD of residential mortgage loans could be affected by an economic cycle. The mean-reverting process has also been adopted for modelling the dynamics of risk-free interest rates with cyclical characteristics. Interest rates appear to be pulled back to some long-run mean level over time (see Vasicek (1977)) according to different states of an economy cycle.³

During an economic recession, many mortgagors cannot afford the repayments of their loans because of wage cut or even unemployment. The default rate would increase to a high level. Banks however do not want to reprocess the properties of defaulted residential mortgage loans, whose values would have been dropped and lower than the outstanding principals of the loans. Some banks would therefore restructure some of their loans such that those mortgagors with financial difficulties could still service the loans. The default rate would not further increase significantly and might come down to a lower mean level when the economy subsequently recovers. Conversely, when the default rate is low during a period of an economic boom including the property market, banks would expand their businesses in property lending. A subsequent economic recession would cause the default rate to increase rapidly from a low level toward a higher mean level. If the above description holds, the stochastic dynamics of the PD could be assumed to be mean-reverting in the model. The PD and the parameters governing its dynamics can be obtained from a bank's historical default data of their retail portfolios.

³ When interest rates are high, the economy tends to slow down and there is less requirement for funds on the part of borrowers. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers. As a result, rates tend to rise.

We derive a closed-form formula from the model as a function of the collateral value and PD to measure the provision of a pool of collateralised retail loans. The two stochastic variables are explicitly correlated in the model. The model allows the PD to follow a mean-reverting process. The model parameters such as, the volatility, correlation and drift of PD are time dependent in the derivation. The model structure, that fits well with the data typically available for banks, can be applied to measuring provisions of retail lending secured by collateral. More accurate measurement of provisioning requirements would enhance banks' capabilities of managing credit risk of such lending.

The scheme of this paper is as follows. In the following section, we develop the model of measuring provisions based on the proposed dynamics of the PD and collateral value. The corresponding closed-form formula is derived. In section 3, we present some empirical findings of the dynamics of the PD of residential mortgage loans and property values based on the residential mortgage market in Hong Kong. The impact of the model parameters on required provisions is studied in section 4. In the last section we shall summarise our investigation.

2. Model for measuring provisions

The PD is defined as an average PD of borrowers in a pool of retail loans over a time horizon of t . The pool is composed of loans with the same collateral type and broadly the same loan-to-value ratio. The value of PD could be an average cumulative default rate over a time horizon (say three years) calculated from a bank's historical data over a period of time (say ten years). Its continuous stochastic movement, which is denoted by D , is governed by a mean-reverting lognormal diffusion process. It follows the stochastic differential equation:

$$\frac{dD}{D} = \kappa_D(t)[\ln \theta_D(t) - \ln D]dt + \sigma_D(t)dz_D. \quad (1)$$

The parameter $\kappa_D(t)$ determines the speed of adjustment toward a mean PD of $\theta_D(t)$. $\sigma_D(t)$ is the volatility of D and dz_D is a standard Wiener process. The model parameters are time dependent. Equation (1) implies that D drifts toward $\theta_D(t)$ when the level of D is different from the mean PD. When $\kappa_D(t)$ is set equal to zero, the dynamics of D is a lognormal diffusion process without any drift.

Let V denote the collateral value securing the loans in the pool. V is assumed to follow a lognormal diffusion process and governed by

$$\frac{dV}{V} = \mu_V(t)dt + \sigma_V(t)dz_V \quad (2)$$

where $\sigma_V(t)$ and $\mu_V(t)$ are the volatility and the rate of return respectively of V . This process is considered to be valid for financial collateral such as equities and physical collateral such as real estate collateral (see for example Kau *et al.* (1992)). The differentials of the Wiener processes dz_D and dz_V in the above equations are correlated with

$$dz_D dz_V = \rho(t)dt. \quad (3)$$

By applying Ito's lemma for equations (1), (2) and (3), the partial differential equation governing the value $P(D, V, t)$ of the provision of the pool of retail loans is

$$\begin{aligned} \frac{\partial P}{\partial t} = & \frac{1}{2}\sigma_D^2(t)D^2 \frac{\partial^2 P}{\partial D^2} + \frac{1}{2}\sigma_V^2(t)V^2 \frac{\partial^2 P}{\partial V^2} + \rho(t)\sigma_D(t)\sigma_V(t)DV \frac{\partial^2 P}{\partial D \partial V} + \\ & [\kappa_D(t)(\ln \theta_D(t) - \ln D)]D \frac{\partial P}{\partial D} + (r - s)V \frac{\partial P}{\partial V} - rP \end{aligned} \quad (4)$$

where r is the risk-free interest rate and s is dividend if the collateral is equities. It is noted that physical collateral (e.g. residential real estate) could be analogous to a stock providing a known dividend yield. The owner of the collateral may receive a yield (e.g. a rental yield) equivalent to a "dividend yield".

The solution of equation (4) is subject to the final condition that represents the loss incurred in a bank over a time horizon of t . The amount of the loss depends on the outstanding loan value L of the loans in the pool and the collateral value after time t . The value L is the bank's actual exposure after taking repayments for the loan principals over time t into account.

As the loans in the pool have broadly the same loan-to-value ratio, the pool can be viewed as an aggregated loan. The final condition of the provision is thus specified as

$$P(D, V, t = 0) = D \max(L - V, 0) \quad (5)$$

where $\max(L - V, 0)$ is the standard payoff of a put option⁴. The solution of equation (4) subject to the final condition (5) is

$$P(D, V, t) = D^{\eta(t)} \exp\left[\int_0^t \alpha(t') dt' - rt\right] \times \left[LN\left(-\frac{z_0}{\sqrt{2c_1}}\right) - V \exp\left[\int_0^t \rho(t') \sigma_D(t') \sigma_V(t') \eta(t') dt' - (s - r)t\right] N\left(-\frac{z_0 + 2c_1}{\sqrt{2c_1}}\right) \right] \quad (6)$$

where

$$\alpha(t) = \left[\kappa_D(t) \ln \theta_D(t) - \frac{\sigma_D^2(t)}{2} \right] \eta(t) + \frac{1}{2} \sigma_D^2(t) \eta^2(t), \quad (7)$$

$$\eta(t) = \exp\left(-\int_0^t \kappa_D(t') dt'\right), \quad (8)$$

$$z_0(V, t) = \ln(V/L) + (r - s)t - c_1(t) + \int_0^t dt' \rho(t') \sigma_D(t') \sigma_V(t') \eta(t'), \quad (9)$$

$$c_1(t) = \int_0^t dt' \frac{\sigma_V^2(t')}{2}, \quad (10)$$

and N is the cumulative normal distribution function.

⁴ Regarding residential mortgage loans with a mortgage-insurance scheme, if an insurance coverage of I due to default is in place, the final condition is modified as $\max(L - V - I, 0)$.

The detailed derivation of the solution in equation (6) is given in the Appendix. When the model parameters are constant or time-dependent functions which can be integrated, $z_0(t)$ and $c_1(t)$ can be integrated analytically and equation (6) is thus in a closed form.

The closed-form solution in equation (6) involves nothing more complex than the standard normal distribution function in terms of D and V . It has an intuitive structure. It is composed of a put-option solution as a function of V multiplied by a factor as a function of D . The put-option part is a decreasing function of the collateral value V and depends on V and L only through their loan-to-value ratio (L/V). The ratio therefore provides a summary measure of facility risk (i.e. LGD) and can be used by banks for categorising internal facility ratings of retail loans. Similarly to a put option, the provision is an increasing function of $\sigma_V(t)$.

The value of D affects the provision of the pool as a multiplication factor with a scale factor of $\eta(t)$ in the solution. The dynamics of D affects the provision indirectly through its correlation ρ with V . A large and negative ρ reduces the value of the second minus term of the solution through $\exp\left[\int_0^t dt' \rho(t') \sigma_D(t') \sigma_V(t') \eta(t')\right]$ and thus increases $P(D, V, t)$. This is consistent with the intuition regarding some retail lending such as residential mortgage loans where the decrease in property prices will increase the default rate of the loans (i.e. $-\rho$), that increases the required provision. Equation (6) also shows that the provision is an increasing function of $\theta_D(t)$. This means that an increase in the mean level of D implies a higher provision of a retail loan portfolio. When the current D is lower (higher) than $\theta_D(t)$, $\kappa_D(t)$ drifts D higher (lower) toward $\theta_D(t)$. A corresponding increase in $\kappa_D(t)$ will increase

(decrease) the provision. If $\kappa_D(t)$ is sufficiently large, D would stick to $\theta_D(t)$ with very small random movement and becomes stationary.

3. Empirical analysis

In this section, we present some empirical findings of the dynamics of the PD of residential mortgage loans and property values based on the data of the residential mortgage market in Hong Kong. The data sample is the monthly problem-loan ratio, which is defined as the sum of the delinquency ratio (i.e. overdue more than three months) and the rescheduled-loan ratio, of residential mortgage loans in banks and the private domestic price index in Hong Kong. The problem-loan ratio can be viewed as a proxy of the default rate of the loans. It is however noted that a one-year default rate is expected to be higher than the problem-loans ratio as the default rate is a cumulative figure while the number of problem loans will be reduced after writing off the loans. The problem-loan ratio and the private domestic price index represent the dynamical variables of the PD (D) and the collateral value (V) specified in equation (1) and (2) respectively. The sample, which covers the periods from June 1998 to February 2004 for D and from January 1993 to February 2004 for V , are published by the Hong Kong Monetary Authority and the Rating and Valuation Department of the Hong Kong SAR Government respectively⁵. This provides with 69 observations for D and 134 observations for V . Using the augmented Dickey-Fuller (ADF) test and the maximum-likelihood technique on the available monthly data, the problem-loan ratio of residential mortgage loans is shown to follow a mean-reverting process. We also describe the econometric approach used for estimating the model parameters relating to D and V .

⁵ The data can be obtained at <http://www.info.gov.hk/hkma/eng/statistics/msb/attach/T0307.xls> and <http://www.info.gov.hk/rvd/property/content.htm>.

Applying Ito's Lemma and defining $X = \ln D$ and $Y = \ln V$, equations (1) and (2) are respectively rewritten as:

$$dX = \kappa_D(t) \left[\ln \theta_D(t) - \frac{1}{2} \sigma_D^2(t) - X \right] dt + \sigma_D(t) dz_D \quad (11)$$

$$dY = \left[\mu_V(t) - \frac{1}{2} \sigma_V^2(t) \right] dt + \sigma_V(t) dz_V. \quad (12)$$

The log values can be characterised by an Ornstein-Uhlenbeck process. Following Brennan and Schwartz (1982), Marsh and Rosenfeld (1983) and Dietrich-Campbell and Schwartz (1986), we can estimate the parameters of the continuous-time model in equations (11) and (12) using the following discrete-time econometric specification:

$$Z_{t+1} - Z_t = \alpha_Z + \beta_Z Z_t + \varepsilon_{Z,t+1} \quad (13)$$

$$E[\varepsilon_{Z,t+1}] = 0, \quad E[\varepsilon_{Z,t+1}^2] = \sigma_Z^2 \quad (14)$$

where Z refers to X or Y . It is worthy mentioning that the econometric specification in equations (13) and (14) assumes that the model parameters are independent of time. This specification is therefore regarded as a time-independent case of the model described in section 2. The corresponding model parameters in equations (1) and (2) can be found by

$$\kappa_D = -\beta_X \quad (15)$$

$$\sigma_D = \sqrt{\sigma_X^2} \quad (16)$$

$$\theta_D = \exp\left(\frac{2\alpha_X + \sigma_X^2}{-2\beta_X}\right) \quad (17)$$

$$\mu_V = \alpha_Y + \frac{1}{2} \sigma_Y^2 \quad (18)$$

$$\sigma_V = \sqrt{\sigma_Y^2} \quad (19)$$

Our econometric approach is to estimate equations (13) and (14) for X (i.e. $\ln D$) and Y (i.e. $\ln V$) using the maximum-likelihood method. The maximum-

likelihood technique has been used in empirical tests of continuous-time models of interest rates by Marsh and Rosenfeld (1983), De Munnik and Schotman (1994) and Bail (1999). In regard of X and Y respectively, we begin by estimating models specified in equations (13) and (14), and assuming $\beta_Y = 0$. These specifications assume that the dynamics of X and Y follow the model description in section 2. Further to this estimation, we consider other model specifications. For X , we consider another specification by restricting the parameters in equation (13) to $\alpha_X = \beta_X = 0$, i.e. no mean-reverting process for X . For Y , we consider a restriction of $\alpha_Y = 0$ in equation (13), i.e. $\mu_Y = \sigma_Y^2 / 2$. The appropriateness of the model restrictions for X and Y is evaluated using the likelihood ratio test.

Table 1 presents the parameter estimates, asymptotic t -statistics, coefficient of determination (R^2) and log-likelihood statistics (LL) for the unrestricted model and the restricted models of X and Y . The likelihood ratio (LR) test statistics of the restricted models are computed to evaluate the appropriateness of the model restriction. For X , the LR test statistic is 81.0620 and the null hypothesis of the test is $\alpha_X = \beta_X = 0$. As the test involves two restrictions on the parameters, the test statistic has a Chi-squared distribution with two degrees of freedom (χ_2^2). As the probability of the χ_2^2 variable being larger than 5.9918 is 5% and the LR test statistic is 81.0620 (>5.9918), the null hypothesis of $\alpha_X = \beta_X = 0$ is rejected at the 5% level of significance. The test result suggests that the unrestricted model for X is preferred and X follows a mean-reverting process. The parameter β_X is negative and implies that κ_D is positive. This finding is consistent with the model description of the PD in section 2.

Further investigation is conducted to determine whether the dynamical process of X is mean-reverting. A unit root test provides a simple method of testing for mean reversion. The unit root test adopted here is the ADF test of Said and Dickey (1984).

Applying the test for X , it requires estimating the following ordinary least squares (OLS) regression equation:

$$X_{t+1} - X_t = \alpha_X + \beta_X X_t + \sum_{i=1}^N \phi_i (X_{t+1-i} - X_{t-i}) + \varepsilon_{X,t+1} \quad (20)$$

The ADF test consists of testing the negativity of β_X in equation (20). To accommodate any serial correlation in the residuals, equation (20) is augmented with N lagged difference terms. The null hypothesis of the ADF test is $\beta_X = 0$ ($\kappa_D = 0$), i.e. X contains a unit root. Rejection of the null hypothesis in favour of the alternative: $\beta_X < 0$ ($\kappa_D > 0$) implies that X is stationary. Stationarity of X implies that the mean and the variance of X are both constant over time.

In order to test the null hypothesis of the ADF test, it is necessary to know the distribution of the statistic used for the test. In this case, it is the distribution of the Dickey-Fuller statistic of β_X , which is derived by the ratio of the OLS estimate of β_X to its OLS standard error. Under the null hypothesis where X contains a unit root, the Dickey-Fuller statistic of β_X does not however have a limiting normal distribution. Dickey and Fuller (1981) show that the distributions of Dickey-Fuller statistics are complicated and, in general, do not have any known analytical forms. It is therefore difficult to find exact critical values for the tests. In practice, critical values are usually approximated through simulation. For example, Cheung and Lai (1995) compute critical values for any sample size and for any number of lagged difference terms using simulation and response surface analysis. The test adopted here is augmented with a drift term (α_X) to use -3.5613 which is given by Cheung and Lai (1995) as the critical values at the 99% confidence level. The ADF test statistic for X

is -6.0470 with one lagged difference term⁶. The result rejects the null of $\beta_X = 0$ (i.e. $\kappa_D = 0$) at the 1% level of significance and thus suggests that X follows a mean-reverting process.

The relative performance of the unrestricted model and the restricted model (i.e. $\alpha_Y = 0$) of Y is evaluated by the LR test statistic. As shown in Table 1, the LR test statistic is 0.2060. The null hypothesis under consideration is $\alpha_Y = 0$. The test here involves a single restriction, and thus the test statistic has a Chi-squared distribution with one degree of freedom (χ_1^2). As the probability of the χ_1^2 variable being larger than 3.8415 is 0.05 and the test statistic is 0.2060 (< 3.8415), we accept the null hypothesis of $\alpha_Y = 0$ at the 5% level of significance. The result indicates that the restricted model is preferred and Y follows an Ornstein-Uhlenbeck process with no mean reversion.

The model parameters relating to D and V can be obtained from the estimates presented in Table 1. Based on the estimation result of the unrestricted model of X and substituting β_X of -0.1499 into equation (15), the parameter κ_D in equation (1) is estimated to be 0.1499. Since the estimation of X is based on the monthly time series, the annualised κ_D is $0.1499 \times 12 = 1.7988$. The long-run PD θ_D can be calculated by substituting the estimates of α_X , β_X and σ_X from the unrestricted model of X into equation (17). The value of θ_D is found to be 0.0144.⁷ The volatility σ_D of D is

⁶ The lag length is chosen using the Akaike Information Criteria and the maximum number of lags is set to be 10.

⁷ The data of the problem-loans ratio (instead of the actual default rate) are used in the estimation. As the number of problem loans will be reduced after writing off the loans, the long-run one-year default rate is expected to be higher than the estimated figure here.

obtained by substituting the estimate of σ_x into equation (16) and σ_D is 0.037. The annualised volatility of D is $0.037 \times \sqrt{12} = 0.1282$.⁸

The drift μ_V and volatility σ_V of V in equation (2) can be estimated by substituting the estimate of σ_Y from the restricted model of Y into equations (18) and (19) respectively. The values of μ_V and σ_V are estimated to be 0.0005 and 0.0314 respectively. The annualised μ_V and σ_V are found to be $0.0005 \times 12 = 0.006$ and $0.0314 \times \sqrt{12} = 0.1087$ respectively.

The value of ρ in equation (3) can be approximated by the sample correlation coefficient between the estimated series of $\varepsilon_{X,t}$ and $\varepsilon_{Y,t}$ in equation (13). The value of the estimated ρ is -0.2603 . The t -statistic⁹ for testing the significance of the estimate is -2.1901 . It is noted that the test statistic has a t -distribution with 66 degrees of freedom (because of the 69 observations for D) and the 5% critical value for a two-tail test is ± 1.9966 . Since the test statistic of -2.1901 is less than -1.9966 , the null hypothesis of no correlation between $\varepsilon_{X,t}$ and $\varepsilon_{Y,t}$ is rejected at the 5% level of significance.

4. Numerical results

The detailed effects of the model parameters on the model provisions required for pools of retail loans are illustrated in the following numerical examples. The current collateral value V of the pool of loans is 1. The time horizon is three years. The average three-year PD of the pool is assumed to be $D = 5\%$ and the mean level

⁸ Based on the data of the property price index during the period from December 1995 to December 1998, its annualised volatility was up to 16.4%.

⁹ The t -statistic is calculated by $t = \rho \sqrt{\frac{n-2}{1-\rho^2}}$. This t -statistic has a student- t distribution with $(n-2)$ degrees of freedom where n is the number of samples.

θ_D of D is set at 8%. The interest rate r and “dividend” yield s are 2.5% per annum. Different values of L are used to illustrate the model provisions for different loan-to-value ratios.

In order to eliminate the effect of the dynamics of D on provisions in Figure 1, that will be studied in Figure 3 below, κ_D and ρ are set to be zero. Figure 1 shows that the increases in provisions depend on the loan-to-value ratio (L/V) and the volatility of the collateral value. For $\sigma_V = 0.1$, the provisions are material when the loan-to-value ratio is larger than 0.8. When σ_V increases to 0.2 and 0.3, the threshold of materiality lowers to about $L/V = 0.6$. The results reflect that provisions are still necessary for loans in positive equity (i.e. $L/V < 1$), in particular where the collateral value is volatile. The increases in provisions with different σ_V are significant when L/V is around 1. Because the model adopts the option-pricing approach, this observation is similar to vega risk of a European option (i.e. sensitivity to the change in volatility of the underlying asset of the option), which is at the maximum when the underlying asset value is equal to the strike price. When L/V is larger than 1.6, the impact of the changes in σ_V is immaterial. This means that the volatility of collateral value does not affect the provisioning requirements for loans which are deep in negative equity.

Based on the model parameters in Figure 1, Figure 2 shows the percentage changes in provisions, which are normalised by dividing by D , with changes in L/V . The results demonstrate that the percentage changes increase with L/V and are at the maximum of 90% when L/V is larger than 1.5. This means that the provisioning requirements for loans in negative equity with $L/V > 1.5$ increase almost linearly with a decline in the collateral value. The provisions for such loans may change rapidly due to the change in the collateral value. Figure 2 also shows that the percentage

changes in provisions increase significantly when the loan-to-value ratios ranges from 0.8 to 1.

Figure 3 illustrates the impact of the mean-reverting process of D and the correlation ρ between D and V on provisions. It considers three cases: (i) $\kappa_D = 0$ and $\sigma_D = 0.22$; (ii) $\kappa_D = 0$ and $\sigma_D = 0.11$; and (iii) $\kappa_D = 0.5$ and $\sigma_D = 0.11$. The volatility σ_V of V is 0.3 per annum. The provision associated a pool with $\rho = 0$ and $\kappa_D = 0$ is 1.1%. The numerical results show that the percentage changes in provisions increase with the decrease (more negative) in the correlation. The negative ρ implies that the PD increases when the collateral value drops. The amount of loss due to default risk and the corresponding provision thus increases. The changes in provisions may be up to 30% in the case of negative ρ , $\kappa_D = 0$ and $\sigma_D = 0.22$. This means that the effect of correlation between PD and the collateral value is material on measuring provisions. Figure 3 shows that the increase in κ_D reduces the changes in provisions with ρ . The positive κ_D implies a stationary movement of D and the effect on changes in provisions due to the correlation is reduced. The result is consistent with the property of the function $z_o(V, t)$ in the solution of equation (6), which shows that the positive κ_D gives a discounting effect on the covariance term $\rho(t)\sigma_D(t)\sigma_V(t)$. The numerical results demonstrate that the effect of an economic cycle (i.e. the present of a mean-reverting process) on the PD could reduce the impact of the correlation on provisions. Figure 3 also shows that the increase in σ_D increases the changes in provisions with ρ , by comparing the results of the cases with $\sigma_D = 0.22$ and 0.11. This observation illustrates that the volatility of the PD may affect provisions through its correlation with the collateral value.

The properties of the required provision-given-default, which is defined as $P(D, V, t)/D$, are illustrated in Figure 4 based on four cases: (i) $\rho = -0.75$ and $\kappa_D = 0$; (ii) $\rho = 0$ and $\kappa_D = 0$; (iii) $\rho = 0.75$ and $\kappa_D = 0$; and (iv) $\rho = 0$ and $\kappa_D = 0.5$. The volatility σ_V and σ_D of V and D are 0.3 and 0.11 per annum respectively. The loan-to-value ratio equal to 1 is used for the calculations. The results show that the provision-given-default increases with time horizons. This property is consistent with the intuition that the uncertainty of the collateral coverage increases with time. When the correlation ρ between D and V is negative and κ_D is equal to zero, the rate of the increments of the provision-given-default with time is higher than the other two cases with higher ρ and zero κ_D . This reflects that the negative correlation between the default rate and collateral value increases the uncertainty of the collateral coverage for loans over time. For the case of $\rho = 0$ and $\kappa_D = 0.5$, as the provision-given-default is based on the current D , the mean-reverting drift of D would pull the current $D = 5\%$ to the mean level of $\theta_D = 8\%$ over time. Where the current PD is lower than the mean-level PD, the mean-reverting drift would increase the provision-given-default with time at a higher rate than the rates observed in the other three cases. The results in Figure 4 demonstrate that collateralised retail loans (in particular long-term lending) require different amounts of provisions for different time horizons.

5. Summary

This paper develops a simple model for measuring the provision for a pool of collateralised retail loans with homogenous characteristics (i.e. the same type of collateral and broadly the same loan-to-value ratio), where the collateral coverage is treated as a put option with the strike price equal to the outstanding loan amount of the pool. The collateral value and the PD of borrowers in the pool are the two

correlated stochastic variables in the model. A closed-form formula of the model is derived and used to calculate the required provision for a pool of loans over a given time horizon. Empirical findings based on the data of the residential mortgage market in Hong Kong are consistent with the proposed mean-reverting dynamics of the PD of residential mortgage loans. The numerical results show that the loan-to-value ratio, correlation between the collateral value and the PD, volatility of the collateral value, mean-reverting process of the PD and time horizon are the important factors for measuring provisions and what are their effects. As the information associated with these factors is in general available in banks' retail portfolios, the model can be readily incorporated into their internal risk management systems as a useful quantitative tool for measuring provisions.

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Appendix

Without loss of generality, the solution $P(D, V, t)$ is rewritten in the form:

$$P(D, V, t) = D^{\eta(t)} F(V, t) \exp\left[\int_0^t \alpha(t') dt'\right], \quad (\text{A.1})$$

where $\alpha(t)$ and $\eta(t)$ are defined in equations (7) and (8) respectively. $F(V, t)$

satisfies the following partial differential equation:

$$\frac{\partial F}{\partial t} = \frac{1}{2} \sigma_V^2(t) V^2 \frac{\partial^2 F}{\partial V^2} + [r - s + \rho(t) \sigma_D(t) \sigma_V(t)] V \frac{\partial F}{\partial V} - rF \quad (\text{A.2})$$

and the corresponding final condition is given by

$$F(V, t = 0) = L \max(1 - V/L, 0). \quad (\text{A.3})$$

It is then not difficult to show that $F(V, t)$ is given by (Lo and Hui, 2001) as

$$\begin{aligned} F(V, t) &= \int_{-\infty}^{\infty} dy K(V, t; y, 0) F(y, 0) \\ &= L \int_{-\infty}^0 dy K(V, t; y, 0) [1 - \exp(y)], \end{aligned} \quad (\text{A.4})$$

where

$$K(V, t; y, 0) = \frac{\exp(-rt)}{\sqrt{4\pi c_1(t)}} \exp\left\{-\frac{[y - z_0(V, t)]^2}{4c_1(t)}\right\} \quad (\text{A.5})$$

is the kernel of equation (A.2), and $z_0(V, t)$ and $c_1(t)$ are defined in equations (9) and (10) respectively. The integral in equation (A.4) can be evaluated analytically to yield a closed-form solution of

$$F(V, t) = L \exp(-rt) \times \left[N\left(-\frac{z_0}{\sqrt{2c_1}}\right) - \exp(z_0 + c_1) N\left(-\frac{z_0 + 2c_1}{\sqrt{2c_1}}\right) \right]. \quad (\text{A.6})$$

After substituting equation (A.6) into equation (A.1), the solution of equation (6) is obtained.

<i>Model (Restriction)</i>	α_X	β_X	σ_X	R^2	LL	LR (<i>p-value</i>)	$\chi^2_{(0.05)}$	df
Unrestricted Model of X	-0.6368 (-11.0417)	-0.1499 (-11.4593)	0.0370 (11.6261)	0.6595	190.315	-	-	-
Restricted Model of X ($\alpha_X = \beta_X = 0$)	-	-	0.0668 (11.7126)	-0.1215	149.784	81.0620 (0.0000)	5.9918	2
	α_Y	β_Y	σ_Y	R^2	LL	LR (<i>p-value</i>)	$\chi^2_{(0.05)}$	Df
Unrestricted Model of Y	-0.0012 (-0.4560)	-	0.0313 (16.2909)	0.0000	394.177	-	-	-
Restricted Model of Y ($\alpha_Y = 0$)	-	-	0.0314 (16.2946)	-0.0016	394.074	0.2060 (0.6499)	3.8415	1

Table 1: This table displays the maximum likelihood estimates of alternative models of X ($\ln D$) and Y ($\ln V$). The parameter estimates with asymptotic t -statistics in parentheses are presented for each model. The maximised log-likelihoods (LL) for the unrestricted models and for each of the restricted models are shown to compare the explanatory power of the unrestricted models and restricted models. Likelihood ratio (LR) tests evaluate the restrictions imposed by the restricted models against the unrestricted model. The LR test statistics with the associated p -value, degrees of freedom (df) and Chi-squared critical values ($\chi^2_{(0.05)}$) at the 5% level of significance are reported. The parameters are estimated from the following discrete time system of equations:

$$Z_{t+1} - Z_t = \alpha_Z + \beta_Z Z_t + \varepsilon_{Z,t+1}$$

$$E[\varepsilon_{Z,t+1}] = 0, \quad E[\varepsilon_{Z,t+1}^2] = \sigma_Z^2$$

where Z refers to X or Y .

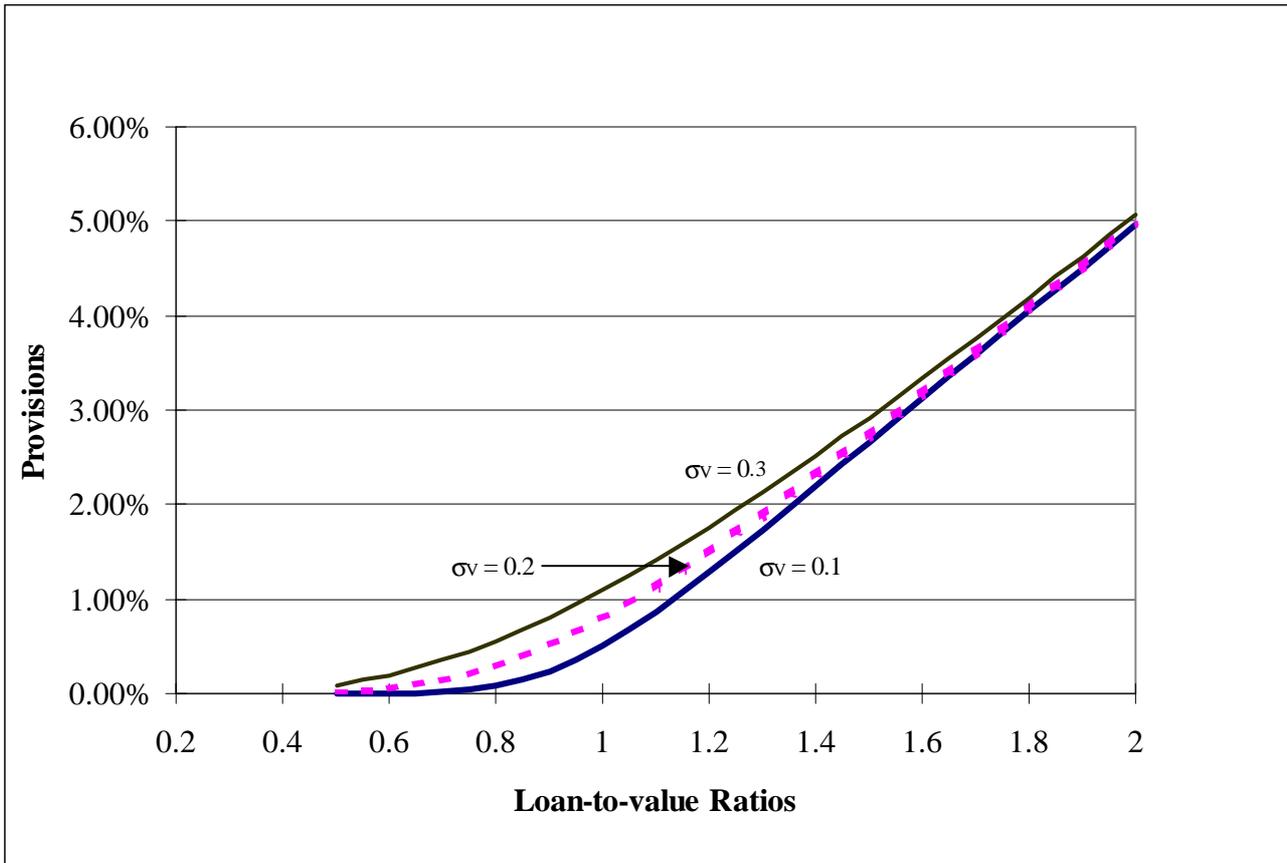


Figure 1. Provisions of pools of loans with different loan-to-value ratios and volatilities of collateral values for a three-year time horizon.

The volatilities of the collateral value are $\sigma_v = 0.1, 0.2$ and 0.3 . The average three-year PD of the pool is 5% with no mean-reverting movement (i.e. $\kappa_D = 0$) and is uncorrelated with the collateral value (i.e. $\rho = 0$). Other parameters are $r = 2.5\%$, $s = 2.5\%$ and $\sigma_D = 0.11$.

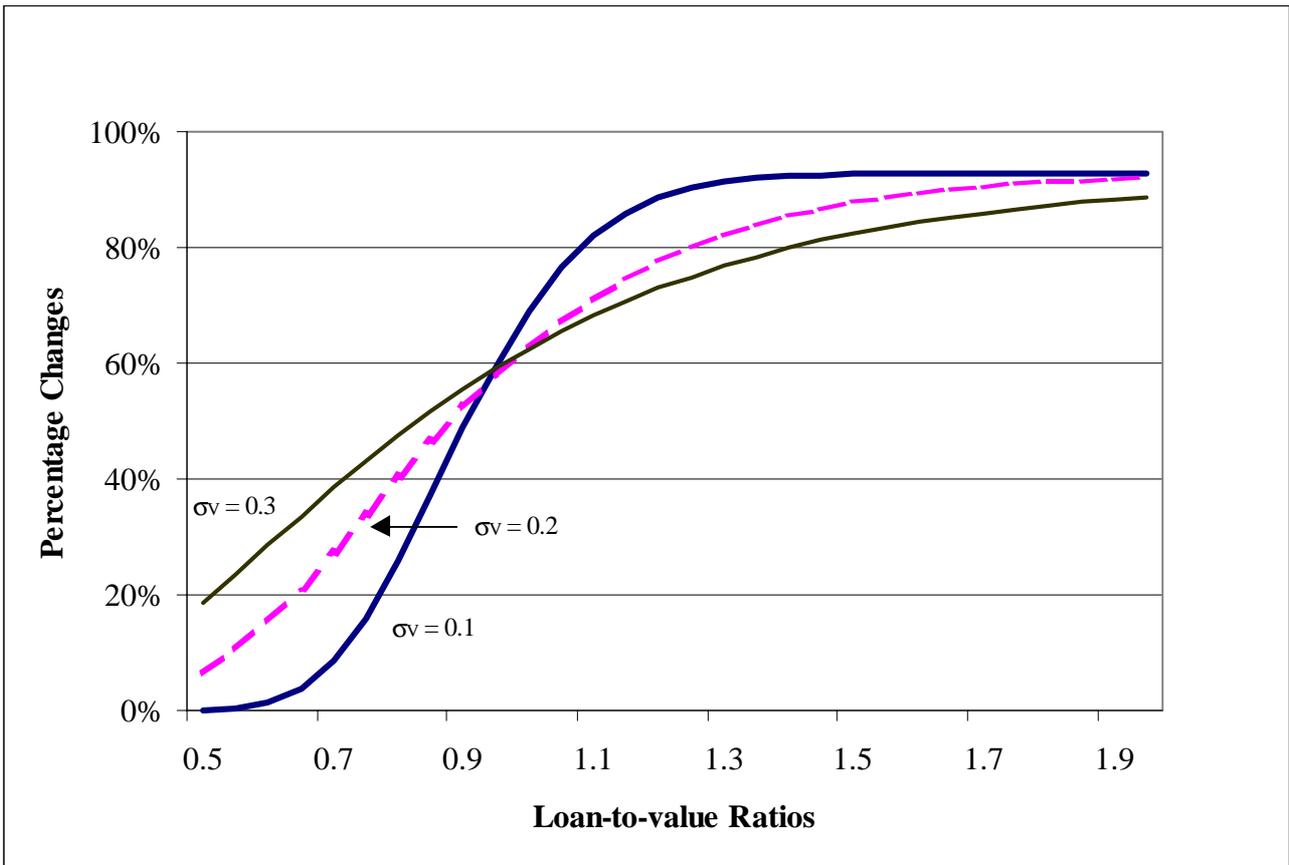


Figure 2. Percentage changes in provisions of pools of loans with different loan-to-value ratios and volatilities of collateral values for a three-year time horizon.

The volatilities of the collateral value are $\sigma_v = 0.1, 0.2$ and 0.3 . The average three-year PD of the pool is 5% with no mean-reverting movement (i.e. $\kappa_D = 0$) and is uncorrelated with the collateral value (i.e. $\rho = 0$). Other parameters are $r = 2.5\%$, $s = 2.5\%$ and $\sigma_D = 0.11$.

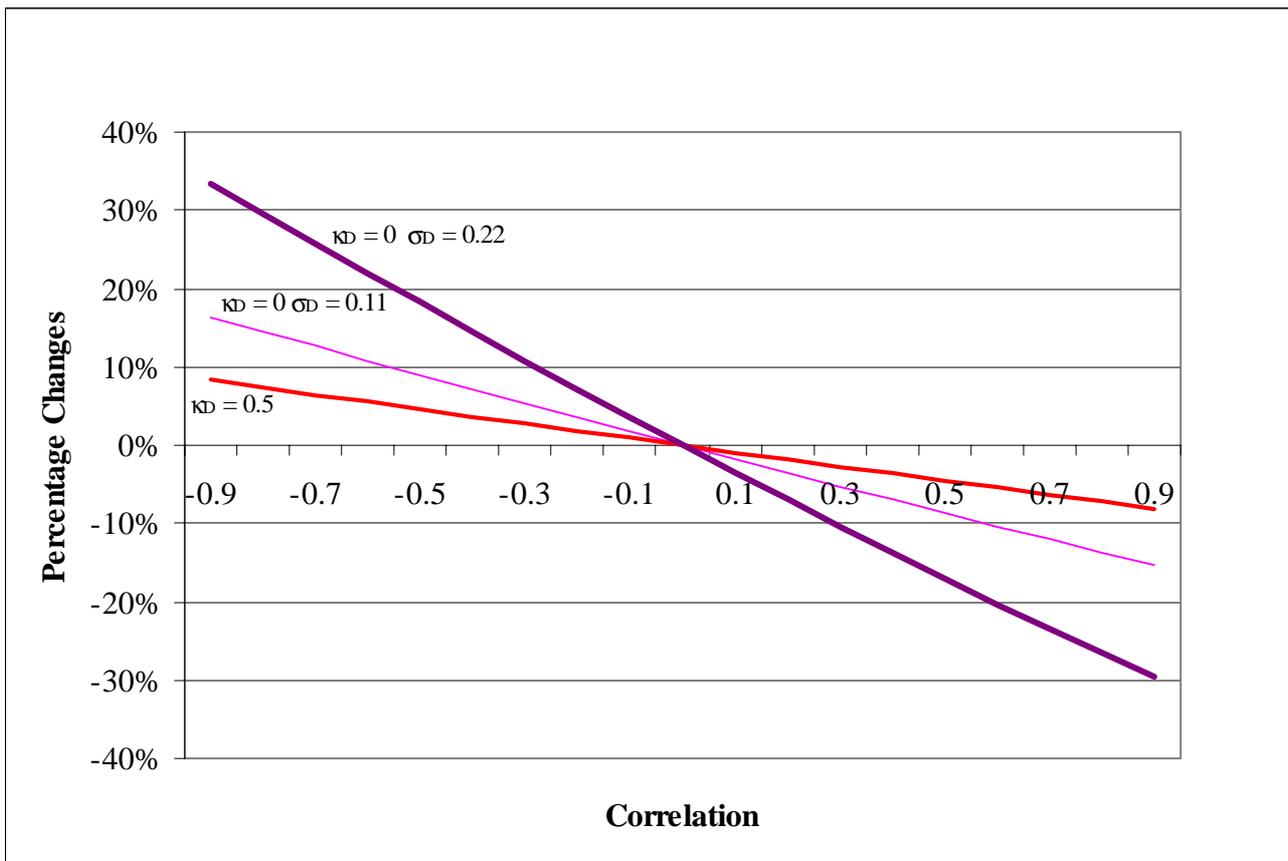


Figure 3. Percentage changes in provisions of pools of loans with different correlation between the collateral value and probability of default under different dynamics of the probability of default.

The loan-to-value ratio (L/V) is 1 and the time horizon is three years. The volatility of the collateral value is $\sigma_V = 0.3$. The dynamics of PD is defined as: (i) $\kappa_D = 0$ and $\sigma_D = 0.22$; (ii) $\kappa_D = 0$ and $\sigma_D = 0.11$; and (iii) $\kappa_D = 0.5$ and $\sigma_D = 0.11$. The average three-year PD of the pool is 5% with the mean level θ_D of D at 8%. Other parameters are $r = 2.5\%$ and $s = 2.5\%$. The provision associated with $\rho = 0$ and $\kappa_D = 0$ is 1.1%.

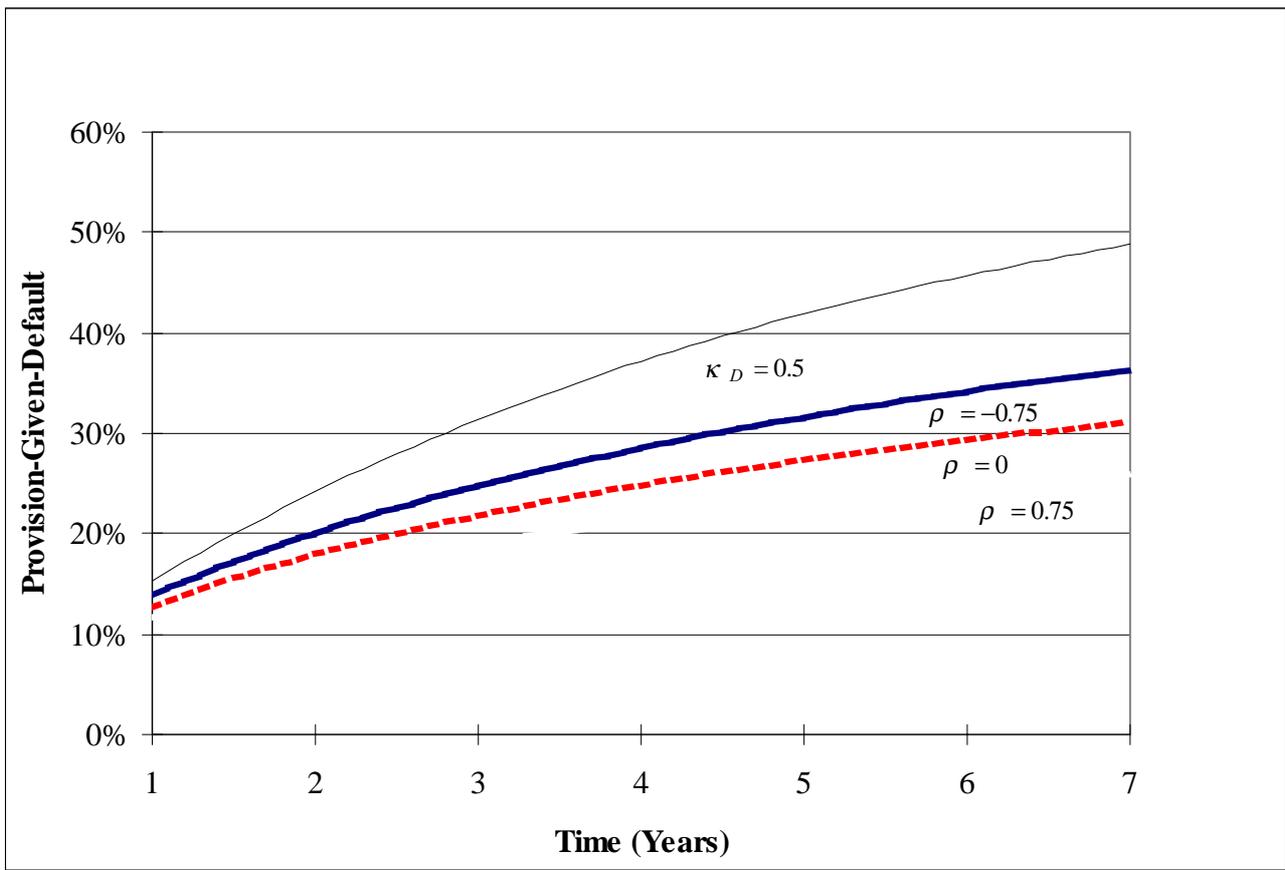


Figure 4. Provision-given-default of pools of loans with different time horizons and correlation between the collateral value and probability of default.

The loan-to-value ratio (L/V) is 1 and the volatility of the collateral value is $\sigma_V = 0.3$.

The four lines represent (i) $\rho = -0.75$ and $\kappa_D = 0$; (ii) $\rho = 0$ and $\kappa_D = 0$; (iii) $\rho = 0.75$ and $\kappa_D = 0$; and (iv) $\rho = 0$ and $\kappa_D = 0.5$. The mean level θ_D of D is at 8%. Other

parameters are $r = 2.5\%$, $s = 2.5\%$ and $\sigma_D = 0.11$.