Two-sided Learning and Optimal Open-Economy Monetary Policy

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Abstract

In this paper, we consider a dynamic New Keynesian model of the small open economy in the light of bounded rationality. This entails private agents and the central bank updating their beliefs about the laws of motion of inflation, the output gap and real exchange rate when forming their optimal decisions. It is shown that a fundamental-shock representation of optimal discretionary policy in the small open economy will yield multiple REE, and in particular, the fundamentals REE cannot be achieved under expectational learning. The alternative representation of optimal policy – an open-economy forecast-based rule – yields a stable fundamentals REE only under certain parameterization when agents learn. Furthermore, the Taylor principle need not be satisfied because part of the stabilization is carried out by the real-exchange-rate channel.

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1 Introduction

The literature on monetary policy thus far has focused on monetary policy design (e.g. Woodford 2003) and interest rate rules (e.g. Taylor 1999) in models where rational expectations equilibrium (REE) as a solution concept is used. The same REE trend is seen in the New Open Economy Macroeconomics (NOEM) literature that discusses monetary policy design and rules for the open economy (e.g. Galí and Monacelli 2002, Clarida, Galí, and Gertler 2001, Monacelli 2003). In this paper, we consider a flexible inflation-targeting central bank in a small open economy where economic agents have to learn about the economy, relaxing the assumption of REE in analyzing monetary policy.

As Evans and Honkapohja (2002) pointed out, two problems emerge when the REE concept is used in the analysis of monetary policy regimes. Firstly, some policy rules can give rise to multiple REE or an indeterminacy problem (e.g. Bernanke and Woodford 1997, Svensson and Woodford 2003). Secondly, Bullard and Mitra (2002) show that when there is learning, the conclusions about alternative policy rules can differ from the outcomes under REE. These two problems suggest that (i) learning can be used as an equilibrium selection device when there exists multiple REE (see e.g. Kirman and Salmon 1995); and (ii) the efficacy of a monetary policy regime depends crucially on expectational dynamics, and therefore, learning has to be taken seriously by policy makers.

A normative question also arises in this context. In some inflation-targeting small open economies (e.g. New Zealand, Canada, United Kingdom), monetary policy is based on explicit forecasts of inflation. However, while Evans and Honkapohja (2003) cite Bank of England and European Central Bank reports which discuss private sector macroeconomic forecasts, they are unsure whether these forecasts are explicitly incorporated into policy making. Should a monetary policy maker respond to forecasts about the state of the economy? This points to the question of whether forecast-based interest rate rules can be stabilizing for an economy. Batini and Haldane (1999) quoted Keynes (1923) who observed,

> If we wait until a price movement is actually afoot before applying remedial measures, we may be too late.

Batini and Haldane (1999) considered interest rules in a small-open-economy model which react to conditional forecasts of inflation at different horizons. In

Batini and Haldane (1999), the analyses of such forecast-based rules are conditioned on REE as a solution concept. However, when such rules give rise to multiple REE or an indeterminacy problem, there may be many or continua of REE, including the fundamentals solution. Thus learning-based expectations can be used as a selection mechanism to justify the stability of a particular REE in the case of indeterminacy. In this paper, the focus will be on whether particular forms of fundamental REE are learnable.[1]

The paper shows that a fundamental-based rule representing optimal discretionary policy in the small open economy yields multiple REE. In particular, the fundamentals form of the REE cannot be achieved under expectational learning. Hence we provide a similar result to Evans and Honkapohja (2002) in this case, and also expanding it to the case of the small open economy. Evans and Honkapohja (2002) also showed that for all parameterization an interest rate rule which responds forecasts of inflation and output gap, in addition to fundamental shocks, can yield a unique fundamental REE under learning. They also showed that such a rule retains the Taylor principle of leaning against the wind, in terms of the interest rate responding more than one-for-one to expected inflation. However, in our small open economy, an alternative representation of optimal policy – an open-economy forecast-based rule – can yield a stable fundamentals REE only under certain parameterization when agents are boundedly rational. Furthermore, the Taylor principle no longer exists in this case. The difference in our result to Evans and Honkapohja (2002) hinges on the additional stabilizing effect of the real exchange rate on the private sector and also on the forecast-based rule.

In terms of the learning approach, we assume that decision makers in the model employ recursive least squares (see e.g. Benveniste, Métivier, and Priouret 1990, Ljung 1977) or a stochastic gradient algorithm (e.g. Sargent 1993, Kuan and White 1994) to update their estimates of their models. This approach, known as sto-

[1] A model is a sequence of probability distributions. In models with rational expectations (RE), it is assumed that decision-making agents know the true probability distributions and the law of motion underlying the contraints they face when they make their best-response decisions. Often, proponents of RE models justify imposing the RE assumption by arguing that optimizing agents have the incentive to revise their subjective beliefs, so that they can be rid of any detectable anomaly between their personal models and what is observed (e.g. Hansen and Sargent 2001). Thus, the rational expectations equilibrium (REE) concept can be thought of as the fixed point of some learning mechanism in which agents' subjective probabilities eventually converge to a model's objective probability.

chastic approximation (see Robbins and Monro 1951) in the engineering or applied mathematics literature, is not of a rational learning nature, but is argued to be a reasonable way of modelling bounded rationality and learning (see Evans and Honkapohja 2002). Sargent (1993) describes this as allowing agents in the model to "behave like econometricians", and not omniscient economic beings.

The rest of the paper is organized as follows. Section 2 provides the description of the structural model. Section 3 analyzes the policy and REE outcome. Section 4 considers the case of optimal discretionary policy and its representation as a fundamentals-based policy rule. Section 4.1 and 4.2 provide the stability results under private-sector and two-sided learning when the policy maker uses an optimal fundamentals-based rule, respectively. Section 5 discusses an alternative representation of optimal policy, in the form of an open-economy forecast-based rule, in both cases of learning. Section 6 concludes.

2 Model

Consider a small open economy described by the following triple (π, x, q), respectively given as the consumer-price-index (CPI) inflation, the output gap, and the real exchange rate. We model these variables as forward-looking in the sense that, *ceteris paribus*, the current values of (π, x, q) will depend on conditional expectations of future realizations of (π, x, q). The "structural form" of the model is as follows:

$$\pi_t = \beta E_t \pi_{t+1} + \lambda x_t + u_t \tag{1}$$

$$x_t = E_t x_{t+1} - \varphi \left(i_t - E_t \pi_{t+1} \right) - \gamma q_t \tag{2}$$

$$q_t = E_t q_{t+1} + i_t - E_t \pi_{t+1} - r_t^* \tag{3}$$

with all variables defined as percentage deviations from their deterministic steady-state values. All parameters are positive except $\gamma < 0$, and the discount factor $\beta \in (0, 1)$. Equation (1) is a familiar New-Keynesian Phillips curve which can be derived from the assumption of monopolistically competitive firms who set prices according to the Calvo (1983) model. Equation (2) is an open-economy IS curve. The positive feedback of the real exchange rate on the output gap is meant to capture the trade-balance effect of real exchange rate movements. Equation (3) is a real interest rate parity condition which says that, all else given, the real return on financial assets across countries are the same in equilibrium. Thus the model is loosely conformable with the models possessing microfoundations of consumers and firms in Monacelli (2003) and Galí and Monacelli (2002). In that case, equations (1)-(3) would represent log-linear approximations of the true nonlinear dynamic optimality conditions of the microfounded model.[2] The nominal interest rate, i_t, is assumed to be controlled by a central bank.

Finally, we assume for the remainder of this paper that r_t^* is given exogenously and we set $r_t^* = r^* = 0$ without harming the rest of our conclusions. This leaves us with one exogenous stochastic process, u_t, in the model. This is interpreted as a cost-push shock in the model. We model this as a first-order Markov process

$$u_t = \rho u_{t-1} + \varepsilon_t \tag{4}$$

where $\rho \in [0, 1)$ and $\varepsilon_t \sim i.i.d.\,(0, \sigma_\varepsilon^2)$.[3]

Notice in equation (1) that with a positive cost-push shock, inflation will rise immediately. Because u_t is persistent when $\rho \neq 0$, expected inflation will be positive as well. Together with a fall in the real exchange rate (equation 3), this results in a fall in the output gap in equation (2), when all else is unchanged. In other words, we have an output-gap-inflation trade-off in the model.

3 Optimal Discretion and Rational Expectations

We take as given, the standard approach in the literature where the central bank is assumed to have a quadratic loss function in output gap and inflation. Under limited assumptions, such a loss function has an approximate mapping from private utilities (e.g. Woodford 2003). In this paper, we take a pragmatic approach, following Jensen (2002), Monacelli (2003) and Svensson (2003), in assuming the given central

[2] The "structural parameters" $(\beta, \lambda, \varphi, \gamma)'$ may in themselves be nonlinear functions of private-sector taste and technology parameters. In this paper, we take these as given and focus on the problem of aggregate private and central-bank learning.

[3] Typically, equation (2) would also be subject to exogenous shocks, interpreted as either government spending or relative technology shocks between the small open economy and the rest of the world. Here, we set this to zero to preserve clarity of analysis.

bank preferences. The central bank's lifetime loss is given by

$$\frac{1}{2}E_t\sum_{s=0}^{\infty}\beta^s\left[(\pi_{t+s}-\overline{\pi})^2+\theta\,(x_{t+s}-\overline{x})^2\right] \tag{5}$$

When $\theta > 0$, this is interpreted as flexible inflation targeting and when $\theta = 0$, we have strict inflation targeting.

In this paper we consider the situation where the central bank acts in discretion. That is the case where the commitment to minimizing (5) subject to (1)-(4) cannot be sustained due to a time-inconsistency problem (e.g. Barro and Gordon 1983, Kydland and Prescott 1977). Subsequently, the central bank decision problem collapses to a period-by-period optimization problem. The relevant solution concept is a Markov perfect equilibrium, where in each period, the decision of the central bank is optimal given the expectations and reactions of the private sector. Implicitly, the private sector is also optimizing given the policy of the central bank.

Note that a positive $\overline{x}$ implies an overambitious central bank and this creates the long run inflation bias problem, as in Barro and Gordon (1983). A positive $\overline{\pi}$ is interpreted as the constraint imposed by the zero lower bound on interest rates. Having positive values for $\overline{\pi}$ and $\overline{x}$ merely introduce intercept terms in the equilibrium solutions. By abstracting from them, one can focus on the short-run stabilization problem. We will assume that $\overline{\pi} = \overline{x} = 0$ for the rest of the paper. The optimization problem of the central bank under discretion is thus given by:

$$\min\frac{1}{2}\left(\pi_{t+s}^2+\theta x_{t+s}^2\right) \tag{6}$$

subject to (1)-(4) for $s = 0, 1, 2, \ldots.$ The necessary first-order condition for a minimum is

$$\theta x_t = -\lambda\pi_t \tag{7}$$

It can be verified that the solution under a rational expectations equilibrium (REE), for some given policy i_t, is linear in the exogenous state variable, u_t. Specifically, it will take the minimal-state-variable (MSV) or fundamentals form

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = \begin{bmatrix} a^{RE} \\ b^{RE} \\ c^{RE} \end{bmatrix} u_t \tag{8}$$

where the parameters $(a, b, c)'$ can be solved using the method of undetermined coefficients. Under rational expectations (RE), we have the conditional projections of the endogenous variables as

$$\begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \\ E_t q_{t+1} \end{bmatrix} = \begin{bmatrix} a^{RE} \\ b^{RE} \\ c^{RE} \end{bmatrix} E_t u_{t+1} = \rho \begin{bmatrix} a^{RE} \\ b^{RE} \\ c^{RE} \end{bmatrix} u_t \tag{9}$$

Making use of (8)-(9) in (1)-(3) and (7), we can solve for the undetermined coefficients as

$$a^{RE} = \frac{\theta}{\theta(1-\beta\rho)+\lambda^2} > 0 \tag{10}$$

$$b^{RE} = -\frac{\lambda}{\theta} a^{RE} = \frac{-\lambda}{\theta(1-\beta\rho)+\lambda^2} < 0 \tag{11}$$

$$\begin{aligned} c^{RE} &= \frac{\lambda\varphi(1-\rho)}{\theta[\varphi(1-\rho)+\gamma]} a^{RE} \\ &= \frac{\lambda\varphi(1-\rho)}{[\varphi(1-\rho)+\gamma]\left[\theta(1-\beta\rho)+\lambda^2\right]} \begin{cases} > 0 \text{ if } |\varphi(1-\rho)| > |\gamma| \\ < 0 \text{ otherwise} \end{cases} . \end{aligned} \tag{12}$$

The optimal policy rule can then be derived by making use of equations (1)-(3) and the optimality condition for the central bank (7) as

$$i_t \equiv \delta_u u_t \tag{13}$$

where

$$\delta_u = \frac{\varphi(1-\rho)[\rho\varphi\theta + \lambda(1-\rho)] + \gamma[\lambda(1-\rho)(1-\varphi) + \rho\varphi\theta]}{\varphi[\varphi(1-\rho)+\gamma]\left[\theta(1-\beta\rho)+\lambda^2\right]}.$$

It can be seen that this model nests the closed-economy case in Evans and Honkapohja (2002).[4] We can substitute out the nominal interest rate from (1)-(3) using (13) to yield the forward-looking trivariate system

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = B \begin{bmatrix} E_t \pi_{t+1} \\ E_t x_{t+1} \\ E_t q_{t+1} \end{bmatrix} + C u_t \tag{14}$$

where

$$B = \begin{bmatrix} \beta + \lambda(\varphi - \gamma) & \lambda & -\lambda\gamma \\ \varphi - \gamma & 1 & -\gamma \\ -1 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 - \delta_u \lambda(\varphi + \gamma) \\ -\delta_u(\varphi + \gamma) \\ \delta_u \end{bmatrix}.$$

It is worth noting that the elasticity δ_u of the optimal interest-rate instrument only affects the variables through the noise term C. The coefficients in B are independent of the policy parameter θ.

4 Instability of Optimal Fundamentals-based Rule

One question is whether the forward-looking system under the fundamentals rule (13) is characterized by a unique and stable REE. This is termed "determinacy" in the REE solution. Using the method of Blanchard and Kahn (1980) or more recently Klein (2000), a determinate solution requires that all the eigenvalues of B lie inside the unit circle. Alternatively, all the eigenvalues of $B - I$ are negative. Intuitively, this ensures that a forward recursive solution of the non-predetermined variables is convergent. If however, there are fewer stable eigenvalues of B than the number of non-predetermined variables, the model is "indeterminate" in the sense that while the MSV form (8) of the solutions exists, other stable solutions which rely on extraneous "sunspot variables" can also exist. This is similar to the fundamentals-based policy rule problem in Sargent and Wallace (1975).[5]

In the following proposition, it is shown that the optimal fundamentals-based

[4] Notice that when $\gamma = 0$, the economy returns to the limit of the two-equation IS-Phillips-curve model in Evans and Honkapohja (2002). Specifically, the optimal fundamentals rule will be $\delta_u = \frac{\rho\varphi\theta + \lambda(1-\rho)}{\varphi[\theta(1-\beta\rho)+\lambda^2]}$, similar to theirs [equation (9)] in terms of the response to the cost-push shock.

[5] See also Svensson and Woodford (2003), Section 2.4.

policy does not yield a stable and unique REE.

Proposition 1 *The economy under (1)-(3), given (4) and (13) is indeterminate for all feasible parameter values.*

The proof to this proposition hinges on the stability of the matrix $B - I$. It can be shown that there are no parameter values such that $|B - I| = 0$ is a stable (third-order) polynomial. This will be discussed at the same time when learning is considered in the next section.

4.1 Instability of REE under Private-sector Learning

In this section, we consider the possibility of relaxing the assumption of model-consistent beliefs on the part of the private sector. Specifically we consider two learning algorithms on the part of the private sector. The central bank, for now, is assumed to know the true model, which is the true sequence of probability distributions governed by the system (1)-(4), for a given interest rate policy. The purpose is to analyze whether the optimal rule (13) in this case will yield a learning equilibrium that asymptotically converges to the MSV or fundamentals REE (8). In this paper, we do not discuss the learnability issue in the case of sunspot equilibria. For learnability of sunspot equilibria in a closed-economy sticky-price monetary model, see Honkapohja and Mitra (2001).

Consider now the transient equilibrium of the model in (1)-(3) which is given by

$$\pi_t = \beta \widetilde{E}_t \pi_{t+1} + \lambda x_t + u_t \tag{1'}$$

$$x_t = \widetilde{E}_t x_{t+1} - \varphi \left(i_t - \widetilde{E}_t \pi_{t+1} \right) - \gamma q_t \tag{2'}$$

$$q_t = \widetilde{E}_t q_{t+1} + i_t - \widetilde{E}_t \pi_{t+1} \tag{3'}$$

and (13). or more compactly we write

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = B \begin{bmatrix} \widetilde{E}_t \pi_{t+1} \\ \widetilde{E}_t x_{t+1} \\ \widetilde{E}_t q_{t+1} \end{bmatrix} + C u_t \tag{15}$$

Agents are still assumed to optimize their underlying objectives, but their beliefs about the future path of the economy are now subjective. Specifically, the one-stepahead conditional expectations of a variable y_t, $\widetilde{E}_t y_{t+1}$, is now calculated based on subjective probability distributions of future variables. In other words, while agents are still learning about the evolution of the economy in order to assess the future path of the economy, the transient equilibrium may be off the REE path. Thus we have to explicitly model how the subjective expectations $\widetilde{E}_t y_{t+1}$ are formed.

Suppose agents have the subjective forecast functions, or perceived law of motion (PLM), given by

$$\begin{bmatrix} \widetilde{E}_t \pi_{t+1} \\ \widetilde{E}_t x_{t+1} \\ \widetilde{E}_t q_{t+1} \end{bmatrix} = \Phi_t + \Psi_t u_t \tag{16}$$

where

$$\Phi_t = \begin{bmatrix} \phi_{1,t} \\ \phi_{2,t} \\ \phi_{3,t} \end{bmatrix}, \Psi_t = \begin{bmatrix} \psi_{1,t} \\ \psi_{2,t} \\ \psi_{3,t} \end{bmatrix} \tag{17}$$

are the periodic coefficients which get updated over time according to some learning rule. We impose a learning rule in the form of a stochastic recursive algorithm.[6] This can either be a recursive least squares (RLS) estimator or a simple stochastic gradient (SG) algorithm. If the REE in (10)-(12) is expectationally stable under our learning scheme, then the estimates in the forecast functions should converge to the RE parameters; $\Phi_t \rightarrow 0$, and $\Psi_t \rightarrow \rho \left(a^{RE}, b^{RE}, c^{RE}\right)'$ in a probabilistic sense.

Given the private sector PLM (16), the law of motion of the economy under learning is

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = B\Phi_t + [B\Psi_t + C] u_t \tag{18}$$

This takes the form of

$$y_t = y\left(\Phi_t, \Psi_t, u_t\right)$$

[6] See Benveniste, Métivier, and Priouret (1990) or Evans and Honkapohja (2001) for the analysis of some adaptive algorithms.

where $y := (\pi, x, q)'$. Let $U_i = (1, u_t)'$. Define the partitions of Φ_t and Ψ_t as:

$$\xi_{1,t} = \begin{bmatrix} \phi_{1,t} \\ \psi_{1,t} \end{bmatrix}, \xi_{2,t} = \begin{bmatrix} \phi_{2,t} \\ \psi_{2,t} \end{bmatrix}, \xi_{3,t} = \begin{bmatrix} \phi_{3,t} \\ \psi_{3,t} \end{bmatrix}$$

as the mean parameter vectors of each equation in (18). Under recursive least squares (RLS) learning, these parameter vectors are updated according to[7]

$$\xi_{1,t} = \xi_{1,t-1} + t^{-1} V_{U,t-1}^{-1} U_{t-1} \left(\pi_{t-1} - \xi'_{1,t-1} U_{t-1} \right) \tag{19}$$

$$\xi_{2,t} = \xi_{2,t-1} + t^{-1} V_{U,t-1}^{-1} U_{t-1} \left(x_{t-1} - \xi'_{2,t-1} U_{t-1} \right) \tag{20}$$

$$\xi_{3,t} = \xi_{3,t-1} + t^{-1} V_{U,t-1}^{-1} U_{t-1} \left(q_{t-1} - \xi'_{3,t-1} U_{t-1} \right) \tag{21}$$

$$V_{U,t} = V_{U,t-1} + t^{-1} \left(U_{t-1} U'_{t-1} - V_{U,t-1} \right). \tag{22}$$

Alternatively, if the learning is done using a stochastic gradient (SG) algorithm, the learning rules become

$$\xi_{1,t} = \xi_{1,t-1} + t^{-1} U_{t-1} \left(\pi_{t-1} - \xi'_{1,t-1} U_{t-1} \right) \tag{23}$$

$$\xi_{2,t} = \xi_{2,t-1} + t^{-1} U_{t-1} \left(x_{t-1} - \xi'_{2,t-1} U_{t-1} \right) \tag{24}$$

$$\xi_{3,t} = \xi_{3,t-1} + t^{-1} U_{t-1} \left(q_{t-1} - \xi'_{3,t-1} U_{t-1} \right). \tag{25}$$

Effectively, the SG algorithm ignores the second moment estimates.

The implied forecast function, or as Evans and Honkapohja (2001) term it, the actual law of motion (ALM) is

$$\begin{bmatrix} \widetilde{E}_t \pi_{t+1} \\ \widetilde{E}_t x_{t+1} \\ \widetilde{E}_t q_{t+1} \end{bmatrix} = B\Phi_t + \rho \left[B\Psi_t + C \right] u_t. \tag{26}$$

Comparing the PLM (16) and the ALM (26), we have a functional of the parameters

[7] Following Evans and Honkapohja (2002), lagged data is used to compute the estimators recursively to avoid issues of simultaneity in the estimation.

(Φ_t, Ψ_t). The functional is given by the map

$$T : \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \to \begin{pmatrix} 0_3 + B\Phi \\ \rho\left[B\Psi + C\right] \end{pmatrix} \tag{27}$$

So an RE solution will be a fixed point $\left(0_3, \Psi^{RE}\right)$ of this mapping. The stability of this RE solution turns out to depend on the stability of the matrix B. This leads us to the following proposition.

Proposition 2 *The REE (8) with (10)-(12) under the optimal fundamental rule (13) is not asymptotically stable for all feasible parameter values, under (i) RLS; or (ii) SG algorithm learning by the private sector.*

Proof. See Appendix B. ■

This result is similar to the strong conclusion in Evans and Honkapohja (2002) that says that with a fundamentals-based policy rule the economy under learning never converges to the REE in a probabilistic sense. Thus, in a small open economy setting with additional learning about the real exchange rate, the fundamentals-based policy rule still does not yield an REE which is stable under learning.

Consider the structure of the model (1')-(3'). Given a positive shock u_t, inflation, π_t, rises, and agents in their PLM (16) will revise their forecast of inflation, $\widetilde{E}_t\pi_{t+1}$, upward, resulting in a rise in x_t and also revision of $\widetilde{E}_t x_{t+1}$ upward in the IS curve. With persistence in u_t, there would be continuous upward revisions of both $\widetilde{E}_t x_{t+1}$ and $\widetilde{E}_t\pi_{t+1}$ in subsequent periods. In this small open economy, there is a further real-exchange-rate channel. Since $\gamma < 0$, learning about the real exchange rate process (3') matters for the stability of $\widetilde{E}_t x_{t+1}$ and $\widetilde{E}_t\pi_{t+1}$as well. Note that, all else equal, the real exchange rate depends on the interest-rate policy. However its positive feedback on x_t and hence π_t in the IS and Phillips curves excarcebate upward movements in $\widetilde{E}_t x_{t+1}$ and $\widetilde{E}_t\pi_{t+1}$ when there is a positive u_t shock. Thus, as in Evans and Honkapohja (2002), a fundamentals based-rule (13) has no means of influencing this upward movement in $\widetilde{E}_t x_{t+1}$ and $\widetilde{E}_t\pi_{t+1}$ thus resulting in the nonconvergence of the economy with learning to the economy with REE.

4.2 Instability of REE under Two-sided Learning

We can also consider generalizing the learning scheme to the case when both private sector and the central bank have to learn about the economy in order to form their optimal decisions. Notice that, for a given temporary equilibrium, (1')-(3'), all private sector agents have to do is periodically estimate the reduced form parameters (Φ_t, Ψ_t) of the model to make a forecast of the endogenous variables. In the case of the central bank with learning, the policy maker has to also estimate the structural parameters in order to set the optimal policy (13). It is assumed that they know the structural form of the economy but not the structural parameters.

Following Evans and Honkapohja (2002), suppose the central bank cannot observe current inflation, output gap and the real exchange rate. This is modelled as an additive uncertainty or "measurement" error for the model. Specifically we have the central bank under discretion now solving:

$$\min \frac{1}{2}\left[\left(\widetilde{E}^{CB}_{t+s}\pi_{t+s}\right)^2 + \theta\left(\widetilde{E}^{CB}_{t+s}x_{t+s}\right)^2\right]$$

subject to

$$\pi_t = \beta \widetilde{E}_t \pi_{t+1} + \lambda x_t + u_t + \varepsilon_{1,t} \tag{28}$$

$$x_t = \widetilde{E}_t x_{t+1} - \varphi\left(i_t - \widetilde{E}_t \pi_{t+1}\right) - \gamma q_t + \varepsilon_{2,t} \tag{29}$$

$$q_t = \widetilde{E}_t q_{t+1} + \left(i_t - \widetilde{E}_t \pi_{t+1}\right) + \varepsilon_{3,t} \tag{30}$$

and (4). The vector of random variables $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)'$ contains white noise. The central bank's optimality condition is now

$$\theta \widetilde{E}^{CB}_t x_t = -\lambda \widetilde{E}^{CB}_t \pi_t \tag{31}$$

It can be shown that the REE under the structural-plus-noise model will given by

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = \begin{bmatrix} a^{RE} \\ b^{RE} \\ c^{RE} \end{bmatrix} u_t + D \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}. \tag{32}$$

where the vector of undetermined coefficients $\left(a^{RE}, b^{RE}, c^{RE}\right)'$ are the same as in

(10)-(12) and

$$D = \begin{bmatrix} 1 & \lambda & -\lambda\gamma \\ 0 & 1 & -\gamma \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider now the case of two-sided learning. We retain the assumption that the central bank can observe the shock u_t and private sector expectations because they know how agents are learning. Assume that the central bank knows θ, its own preferences, and the discount factor β is commonly known. It is assumed also that the central bank is also uncertain as to the elasticity of expected exchange rate growth with respect to real interest rates in the estimated version of (30). Thus we have

$$q_t = \widetilde{E}_t q_{t+1} + \chi_t \left(i_t - \widetilde{E}_t \pi_{t+1} \right) + \varepsilon_{3,t} \tag{33}$$

so that χ is introduced as another parameter to be estimated.

All the central bank has to estimate then are the structural parameters $(\lambda, \varphi, \gamma, \chi)$ in the structural-plus-noise model above. Define the following data series:

$$z_{1,t} = \pi_t - \beta \widetilde{E}_t \pi_{t+1} - u_t$$

$$z_{2,t} = x_t - \widetilde{E}_t x_{t+1}$$

$$z_{3,t} = q_t - \widetilde{E}_t q_{t+1}$$

$$r_t = i_t - \widetilde{E}_t \pi_{t+1}$$

so that the central bank estimates the following equations

$$z_{1,t} = \lambda x_t + \varepsilon_{1,t} \tag{34}$$

$$z_{2,t} = -\varphi r_t - \gamma q_t + \varepsilon_{2,t} \tag{35}$$

$$z_{3,t} = \chi r_t + \varepsilon_{3,t}. \tag{36}$$

The implied optimal policy rule is

$$i_t = \frac{\widehat{\varphi}_t (1-\rho) \left[\rho \widehat{\varphi}_t \theta + \widehat{\lambda}_t (1-\rho)\right] + \widehat{\gamma}_t \left[\widehat{\lambda}_t (1-\rho)(1-\widehat{\varphi}_t) + \rho \widehat{\varphi}_t \theta\right]}{\widehat{\varphi}_t \left[\widehat{\varphi}_t (1-\rho) + \widehat{\gamma}_t\right] \left[\theta (1-\beta\rho) + \widehat{\lambda}_t^2\right]} u_t \equiv \widehat{\psi}_{u,t} u_t \quad (37)$$

where now the parameters with a "hat" are estimated by the central bank by either RLS or the SG algorithm. The RLS estimators for the central bank are given by

$$\widehat{\lambda}_t = \widehat{\lambda}_{t-1} + t^{-1} \widehat{R}_{x,t-1}^{-1} x_{t-1} \left(z_{1,t-1} - \widehat{\lambda}_{t-1} x_{t-1}\right)$$

$$\widehat{R}_{x,t} = \widehat{R}_{x,t-1} + t^{-1} \left(x_{t-1}^2 - \widehat{R}_{x,t-1}\right)$$

$$\begin{bmatrix} \widehat{\varphi}_t \\ \widehat{\gamma}_t \end{bmatrix} = \begin{bmatrix} \widehat{\varphi}_{t-1} \\ \widehat{\gamma}_{t-1} \end{bmatrix} + t^{-1} R_{rq,t-1}^{-1} \begin{bmatrix} r_{t-1} \\ q_{t-1} \end{bmatrix} \left(z_{2,t-1} - \begin{bmatrix} r_{t-1} \\ q_{t-1} \end{bmatrix}' \begin{bmatrix} \widehat{\varphi}_{t-1} \\ \widehat{\gamma}_{t-1} \end{bmatrix} \right)$$

$$\widehat{R}_{rq,t} = \widehat{R}_{rq,t-1} + t^{-1} \left(\begin{bmatrix} r_{t-1} \\ q_{t-1} \end{bmatrix} \begin{bmatrix} r_{t-1} \\ q_{t-1} \end{bmatrix}' - \widehat{R}_{rq,t-1} \right)$$

$$\widehat{\chi}_t = \widehat{\chi}_{t-1} + t^{-1} \widehat{R}_{r,t-1}^{-1} x_{t-1} \left(z_{3,t-1} - \widehat{\chi}_{t-1} r_{t-1}\right)$$

$$\widehat{R}_{r,t} = \widehat{R}_{r,t-1} + t^{-1} \left(r_{t-1}^2 - \widehat{R}_{r,t-1}\right)$$

In the case of learning using a SG algorithm, we can just ignore the second-moment estimation equations. Given the private sector PLM (16), the law of motion of the economy under learning is now

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = B\Phi_t + \left[B\Psi_t + C\right] u_t + D\varepsilon_t$$

This takes the form of

$$y_t = y\left(\Phi_t, \Psi_t, u_t, \varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}\right)$$

where $y := (\pi, x, q)'$. The private sector estimates are still recursively updated as in (19)-(22) or (23)-(25). For the central bank we have the data generating mechanism under learning as

$$y_t = y\left(\widehat{\lambda}_t, \widehat{\varphi}_t, \widehat{\gamma}_t, \widehat{\chi}_t, \Phi_t, \Psi_t, u_t, \varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}\right)$$

if we make use of the definitions of variable (z_1, z_2, z_3) above, the private agents' PLM (16), and the subjectively optimal rule (37).

We arrive at the following result for learning by both private sector and central bankers.

Proposition 3 *The small open economy converges to the REE in (32) when both private sector and central bank are learning with RLS or a SG algorithm with probability zero for all parameter values, when the subjectively optimal rule is (37).*

Proof. See Appendix C. ■

The intuition for this result is as follows. As long as the central bank's estimates of structural parameters converge to the true REE parameters under learning, the subjectively optimal rule (37) will converge to the same rule under REE (13). The asymptotically convergent central bank estimates also affect private sector subjective forecasts of the economy. Given that the central bank rule under asymptotic learning is the REE rule, then the private sector reduced-form learning is also convergent in a probabilistic sense, as in Proposition 2. However, in this case, the private sector learning is nonconvergent to the REE resulting in nonconvergence of the central bank rule to REE because the central bank makes use of private sector expectations in estimating its structural parameters.

5 Stability of Optimal Forecast-Based Rule

In this section, we focus on the case of private-sector and two-sided learning, where the central bank's optimality condition is still the same as (7) or (31), respectively. However, an alternative expression for the implied optimal interest rate can be derived. It is shown that when the central bank takes advantage of observed private sector forecasts of inflation, output gap and the real exchange rate in its optimal policy rule, the economy under learning by the private sector or both private sector and policy maker is expectationally stable.

5.1 Stability Result for Private-Sector Learning

Suppose again, that the private sector has to learn about the state of the economy and forecast using RLS or SG algorithms, but the central bank sets policy optimally as if under RE using (1)-(3) and (13). An alternative expression for the optimal policy rule is

$$i_t = \alpha_\pi \widetilde{E}_t \pi_{t+1} + \alpha_x \widetilde{E}_t x_{t+1} + \alpha_q \widetilde{E}_t q_{t+1} + \alpha_u u_t \tag{38}$$

where

$$\begin{aligned} \alpha_x &= (\varphi + \gamma)^{-1} \\ \alpha_\pi &= 1 + \frac{\lambda \beta \alpha_x}{\lambda^2 + \theta} \\ \alpha_q &= \gamma \alpha_x \\ \alpha_u &= \frac{\gamma \alpha_x}{\lambda^2 + \theta}. \end{aligned}$$

As in Evans and Honkapohja (2002), this policy implements the optimal discretionary policy every period regardless of whether private expectations of inflation, output gap or real exchange rate are in or out of an REE. That is the rule sets interest rates in response to shifts in private forecasts. We shall denote such rules as optimal forecast-based rules.

The reduced form of the model is now

$$\begin{bmatrix} \pi_t \\ x_t \\ q_t \end{bmatrix} = \begin{bmatrix} \frac{\beta\theta}{\lambda^2+\theta} & 0 & 0 \\ \frac{-\lambda\beta}{\lambda^2+\theta} & 0 & 0 \\ \frac{\lambda\beta}{(\varphi+\gamma)(\lambda^2+\theta)} & \frac{1}{(\varphi+\gamma)} & \frac{\varphi}{(\varphi+\gamma)} \end{bmatrix} \begin{bmatrix} \widetilde{E}_t \pi_{t+1} \\ \widetilde{E}_t x_{t+1} \\ \widetilde{E}_t q_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{\theta}{\lambda^2+\theta} \\ \frac{-\lambda}{\lambda^2+\theta} \\ \frac{\lambda}{(\varphi+\gamma)(\lambda^2+\theta)} \end{bmatrix} u_t + D \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \tag{39}$$

where D is the same as before.

Proposition 4 *Suppose the central bank sets optimal policy using the forecast-based rule (38), and the private sector learns using either (i) RLS or (ii) a SG algorithm. The transient equilibrium (1'), (2') and (3'), given (4) converges almost surely to the REE (8) if and only if in the true model,* $|\gamma| > |\varphi|$.

Proof. See Appendix D. ■

5.2 Stability Result for Two-Sided Learning

Consider now the case where the central bank has to learn about the structural parameters. Here, the central bank makes use of the first-order condition (31) in (28), (29) and (30). The forecast-based policy rule becomes

$$i_t = \widehat{\alpha}_{\pi,t}\widetilde{E}_t\pi_{t+1} + \widehat{\alpha}_{x,t}\widetilde{E}_t x_{t+1} + \widehat{\alpha}_{q,t}\widetilde{E}_t q_{t+1} + \widehat{\alpha}_{u,t}u_t \tag{40}$$

where

$$\begin{aligned}
\widehat{\alpha}_{x,t} &= (\widehat{\varphi}_t + \widehat{\gamma}_t)^{-1} \\
\widehat{\alpha}_{\pi,t} &= 1 + \frac{\beta\widehat{\lambda}_t\widehat{\alpha}_{x,t}}{\widehat{\lambda}_t^2 + \theta} \\
\widehat{\alpha}_{q,t} &= \widehat{\gamma}_t\widehat{\alpha}_{x,t} \\
\widehat{\alpha}_{u,t} &= \frac{\widehat{\lambda}_t\widehat{\alpha}_{x,t}}{\widehat{\lambda}_t^2 + \theta}
\end{aligned}$$

Conditional on the estimates in each period of $\left(\widehat{\lambda}_t, \widehat{\varphi}_t, \widehat{\gamma}_t, \widehat{\chi}_t\right)'$ and the private sector forecasts, the transient equilibrium is now governed by (28), (29) and (30), given (4) and (40).

Proposition 5 *Suppose the central bank sets optimal policy using the forecast-based rule (40), and both central bank and private sector are learning using either (i) RLS or (ii) a SG algorithm. The transient equilibrium (28), (29) and (30), given (4) converges almost surely to the REE (32) if and only if in the true model, $|\gamma| > |\varphi|$.*

Proof. The proof of this proposition is straightforward. Appendix C shows that for central-bank learning, using (34)-(36), $\left(\widehat{\lambda}_t, \widehat{\varphi}_t, \widehat{\gamma}_t, \widehat{\chi}_t\right) \rightarrow w.p.1\,(\lambda, \varphi, \gamma, 1)$ and so the forecast-based rule (40) converges with probability one to (38). Given this condition, Proposition 4 applies. ∎

If $|\gamma| > |\varphi|$, this implies that $\alpha_x < 0$, $\alpha_\pi < 1$ and $\alpha_q > 0$. A partial equilibrium intuition for Propositions 4 and 5 is as follows. Consider a rise in $\widetilde{E}_t\pi_{t+1}$. Assuming $0 < \alpha_\pi < 1$, the interest rate will only rise partially in direct response to $\widetilde{E}_t\pi_{t+1}$ resulting in a decline of the *ex-ante* real interest rate which pressures x_t to rise via φ in (29). However, the tendency for x_t to rise is countervailed by the fall in q_t in (3) in response to $\widetilde{E}_t\pi_{t+1}$ rising, and this feeds back into the IS curve (29) via γ. Thus, even without a "lean against the wind" response to inflation expectations ($\alpha_\pi < 1$) as Evans and Honkapohja (2002) suggest in their version of the Taylor principle, in this small open economy part of the stabilization effect of the forecast-based rule comes from the real exchange rate channel via the trade balance.

Similarly, consider a partial argument where $\widetilde{E}_t q_{t+1}$ has increased. This would cause q_t to rise, all else equal, and thus x_t to rise via γ in the IS curve (29). Again, the forecast based rule would respond to this by raising the interest rate since $\alpha_q > 0$. This acts to lower and thus stabilize x_t via φ in the IS curve.

Finally, consider a higher $\widetilde{E}_t x_{t+1}$. This directly increases x_t in the IS curve. First, the forecast-based rule would decrease interest rate since $\alpha_x < 0$, and this would tend to push x_t up. This tends to decrease q_t and thus x_t, resulting in its stabilization.

For a complete view on the stabilizing effect of the forecast-based rule under learning, when $|\gamma| > |\varphi|$, we will have to simulate the model using a calibrated example. The dynamic adjustment path now will also be complicated by real-time learning dynamics.

5.3 Discussion

An interesting policy problem arises in this context. As noted by Evans and Honkapohja (2002), to implement the policy in (38) or (40), the central bank needs to know how the private sector makes its forecasts of key macroeconomic variables. In our case, we have the additional requirement that the central bank observes private expectations of the real exchange rate. In this model, the observability of private forecasts is straightforward since the central bank knows that private agents are forecasting using a correctly identified vector autoregressive (VAR) model.

In more realistic problems, the private sector may not know the true solution underlying the economy in the expectational limit of an REE. Thus, they may be forecasting using misspecified models. In fact Friedman (1959) had warned:

> Leaning today against next year's wind is hardly an easy task in the present state of meteorology.

This issue is not discussed in this paper but will be addressed in future research. Another related issue is whether the central bank can accurately observe private expectations, even if the private sector uses a correct form of the forecasting model. The conjecture is that it will be a straightforward signal processing problem, especially if measured expectations are merely contaminated by white noise. See Evans and Honkapohja (2002).

6 Conclusion

In this paper a New Keynesian small open economy model with optimal monetary policy is considered. We analyze the stability of the rational expectations equilibrium (REE) under the cases of private-sector learning and simultaneous private-sector and central-bank learning when the central bank acts in discretion. The implied optimal policy rule in these cases is a rule that reacts to fundamental cost-push shocks. Alternative, one can represent the same optimal policy as an open-economy forecast-based rule, where the interest rate is set in response to subjective private sector forecasts of inflation, output gap, real exchange rate and fundamental shocks.

The paper shows that a fundamental-shock representation of optimal discretionary policy in the small open economy will yield multiple REE. Specifically the fundamentals form of the REE cannot be achieved under expectational learning when monetary policy responds only to the exogenous shocks. The alternative representation of optimal policy – an open-economy forecast-based rule – yields a stable fundamentals REE only under certain parameterization when agents learn. Furthermore, the Taylor principle need not be satisfied in this case because the endogenous real-exchange-rate channel on the private sector also acts to stabilize the economy.

In terms of policy prescription, if policy makers in small open economies take expectational learning seriously in setting monetary policy, then an appropriately-designed policy which reacts to accurately-measured private sentiments of inflation, activity and the real exchange rate, can still deliver a stable fundamental REE in an open economy.

The subject of whether the central bank can measure private expectations accurately, or even if they can, whether the private sector forecasts correctly is interesting. These questions are explored in Evans and Honkapohja (2002) and Evans and Honkapohja (2001), respectively. It would be straightforward in the example of observation of expectations with white-noise errors, as in Evans and Honkapohja (2002). Evans and Honkapohja (2001) discuss misspecified perceived laws of motion in other types of models.

Appendix

A Stable Polynomials

The well-known Routh-Hurwitz theorem is required for proving Propositions 2 and 3:[8]

Lemma A.1 *(Routh-Hurwitz Theorem) For a polynomial* $\alpha_0\lambda^n + \alpha_1\lambda^{n-1} + ... + \alpha_{n-1}\lambda + \alpha_n = 0$ *to be stable, the necessary and sufficient condition is that all roots to the polynomial have negative real parts. The latter holds if and only if*

$$\alpha_1 > 0, \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} > 0, \begin{vmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} > 0, ..., \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 & \cdots & \cdots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \cdots & \cdots \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_n \end{vmatrix} > 0.$$

B Proof of Proposition 2

Here, we construct the proof of Proposition 2. Consider the optimal fundamentals-based policy rule (13). Under this rule, the transient equilibrium is given by (15). Define the parameter vector $\Theta_t = \left(\xi'_{1,t}, \xi'_{2,t}, \xi'_{3,t}, \text{vec}\left(V_{U,t}\right)'\right)'$ and state vector $X_t = (u_t, u_{t-1})$. The RLS or decreasing-gain learning rules have the form of a stochastic approximation algorithm given by

$$\Theta_t = \Theta_{t-1} + t^{-1}\mathcal{H}\left(\Theta_{t-1}, X_t\right) \tag{41}$$

where $\mathcal{H}\left(\Theta_{t-1}, X_t\right)$ is a vector field generated by the stochastic process X_t. For every Θ_{t-1} fixed at some Θ, we calculate the mean field

$$h\left(\Theta\right) = \lim_{t\to\infty} E\mathcal{H}\left(\Theta, X_t\right)$$

Therefore $h\left(\Theta\right)$ refers to the asymptotic law where X_t is stationary.

[8] See for example (p.343, Takayama 1994) and Holtz (2003).

Under Assumptions A, B and the subsequent results in Chapter 6 of Evans and Honkapohja (2001), convergence of the algorithm (41) is governed by the stability of the associated ordinary equation (ODE)

$$\frac{d\Theta}{d\tau} = h(\Theta).$$

where τ is notional time. Under one-sided private sector learning, we have

$$\frac{d\xi_1}{d\tau} = \begin{bmatrix} \frac{d\phi_1}{d\tau} \\ \frac{d\psi_1}{d\tau} \end{bmatrix} = V_U^{-1} E\left(U_{t-1}U_{t-1}'\right) \left\{ \begin{bmatrix} (\beta + \lambda(\varphi - \gamma))\phi_1 + \lambda\phi_2 - \lambda\gamma\phi_3 \\ \rho(+(\beta + \lambda(\varphi - \gamma))\psi_1 + \lambda\psi_2 - \lambda\gamma\psi_3 + 1 - \lambda\delta_u(\varphi + \gamma)) \end{bmatrix} - \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \right\} \tag{42}$$

$$\frac{d\xi_2}{d\tau} = \begin{bmatrix} \frac{d\phi_2}{d\tau} \\ \frac{d\psi_2}{d\tau} \end{bmatrix} = V_U^{-1} E\left(U_{t-1}U_{t-1}'\right) \left\{ \begin{bmatrix} \phi_1(\varphi - \gamma) + \phi_2 - \gamma\phi_3 \\ \rho((\varphi - \gamma)\psi_1 + \psi_2 - \gamma\psi_3 - \delta_u(\varphi + \gamma)) \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \right\} \tag{43}$$

$$\frac{d\xi_3}{d\tau} = \begin{bmatrix} \frac{d\phi_3}{d\tau} \\ \frac{d\psi_3}{d\tau} \end{bmatrix} = V_U^{-1} E\left(U_{t-1}U_{t-1}'\right) \left\{ \begin{bmatrix} -\phi_1 + \phi_3 \\ \rho(-\psi_1 + \psi_3 + \delta_u) \end{bmatrix} - \begin{bmatrix} \phi_3 \\ \psi_3 \end{bmatrix} \right\} \tag{44}$$

$$\frac{dV_U}{d\tau} = E\left(U_{t-1}U_{t-1}'\right) - V_U(\tau) \tag{45}$$

Firstly, the ODE governing the second moment estimates (45) is stable since $V_U(\tau) \to E\left(UU'\right)$. So local stability of the REE will depend on the stability of the smaller ODE

$$\frac{d\xi_1}{d\tau} = \begin{bmatrix} (\beta + \lambda(\varphi - \gamma))\phi_1 + \lambda\phi_2 - \lambda\gamma\phi_3 \\ \rho(+(\beta + \lambda(\varphi - \gamma))\psi_1 + \lambda\psi_2 - \lambda\gamma\psi_3 + 1 - \lambda\delta_u(\varphi + \gamma)) \end{bmatrix} - \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}$$

$$\frac{d\xi_2}{d\tau} = \begin{bmatrix} \phi_1(\varphi - \gamma) + \phi_2 - \gamma\phi_3 \\ \rho((\varphi - \gamma)\psi_1 + \psi_2 - \gamma\psi_3 - \delta_u(\varphi + \gamma)) \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}$$

$$\frac{d\xi_3}{d\tau} = \begin{bmatrix} -\phi_1 + \phi_3 \\ \rho\left(-\psi_1 + \psi_3 + \delta_u\right) \end{bmatrix} - \begin{bmatrix} \phi_3 \\ \psi_3 \end{bmatrix}$$

This system can be re-stacked to yield

$$\begin{bmatrix} \frac{d\phi_1}{d\tau} \\ \frac{d\phi_2}{d\tau} \\ \frac{d\phi_3}{d\tau} \end{bmatrix} = [B - I_3] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \tag{46}$$

$$\begin{bmatrix} \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\tau} \\ \frac{d\psi_3}{d\tau} \end{bmatrix} = \rho\,[B - I_3] \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} + \rho C \tag{47}$$

which is the functional as heuristically given in (27). If we replace the RLS algorithm with a simple decreasing-gain algorithm the associated ODE is still (46)-(47).

The stability of (46)-(47) depends on the eigenvalues of the matrix $B - I_3$. The characteristic polynomial for this matrix is

$$P(\epsilon) = \epsilon^3 + [1 - \beta - \lambda(\varphi - \gamma)]\,\epsilon^2 - \lambda\varphi\epsilon - \lambda\gamma = 0$$

where ϵ is an eigenvalue that solves $P(\epsilon) = 0$.

Suppose the polynomial $P(\epsilon) = 0$ is stable such that all the following conditions from Lemma A.1 hold:

1. $1 - \beta - \lambda(\varphi - \gamma) > 0 \Leftrightarrow \varphi - \gamma < \frac{1-\beta}{\lambda}$
2. $-\lambda\varphi > 0$
3. $-\lambda\gamma > 0$
4. $\varphi - \gamma \in \left(\frac{1-\beta+\lambda\gamma-\sqrt{(1-\beta+\lambda\gamma)^2+4\beta\lambda\gamma}}{2\lambda}, \frac{1-\beta+\lambda\gamma+\sqrt{(1-\beta+\lambda\gamma)^2+4\beta\lambda\gamma}}{2\lambda} \right)$.

These inequalities imply:

1. $\varphi \in (\underline{a}, \overline{a})$ where $\underline{a} = \frac{1-\beta+3\lambda\gamma-\sqrt{(1-\beta+\lambda\gamma)^2+4\beta\lambda\gamma}}{2\lambda}$ and $\overline{a} = \frac{1-\beta+3\lambda\gamma+\sqrt{(1-\beta+\lambda\gamma)^2+4\beta\lambda\gamma}}{2\lambda}$ or $\frac{1-\beta+\gamma\lambda}{\lambda}$.
2. $-\lambda\varphi > 0$

3. $-\lambda\gamma > 0$

Since all structural parameters must be positive except $\gamma < 0$, Condition 2 is a contradiction. Hence $B - I_3$ is unstable for all feasible parameter values.

C Proof of Proposition 3

Let the $\widetilde{\Theta}_t = \left(\widehat{\lambda}_t, \widehat{\varphi}_t, \widehat{\gamma}_t, \widehat{\chi}_t, \Phi_t, \Psi_t, u_t, \varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}\right)$. The associated ODE for the central bank's RLS estimators are

$$\frac{d\widehat{\lambda}}{d\tau} = R_x^{-1} Ex\left(\widetilde{\Theta}_{t-1}\right)^2 \left(\lambda - \widehat{\lambda}\right)$$

$$\frac{dR_x}{d\tau} = Ex\left(\widetilde{\Theta}_{t-1}\right)^2 - R_x$$

$$\begin{bmatrix} \frac{d\widehat{\varphi}}{d\tau} \\ \frac{d\widehat{\gamma}}{d\tau} \end{bmatrix} = R_{rq}^{-1} E\left\{ \begin{bmatrix} r\left(\widetilde{\Theta}_{t-1}\right) \\ q\left(\widetilde{\Theta}_{t-1}\right) \end{bmatrix} \begin{bmatrix} r\left(\widetilde{\Theta}_{t-1}\right) \\ q\left(\widetilde{\Theta}_{t-1}\right) \end{bmatrix}' \right\} \left(\begin{bmatrix} \varphi \\ \gamma \end{bmatrix} - \begin{bmatrix} \widehat{\varphi} \\ \widehat{\gamma} \end{bmatrix} \right)$$

$$\frac{dR_{rq}}{d\tau} = E\left\{ \begin{bmatrix} r\left(\widetilde{\Theta}_{t-1}\right) \\ q\left(\widetilde{\Theta}_{t-1}\right) \end{bmatrix} \begin{bmatrix} r\left(\widetilde{\Theta}_{t-1}\right) \\ q\left(\widetilde{\Theta}_{t-1}\right) \end{bmatrix}' \right\} - R_{rq}$$

$$\frac{d\widehat{\chi}}{d\tau} = R_r^{-1} Er\left(\widetilde{\Theta}_{t-1}\right)^2 \left(1 - \widehat{\chi}\right)$$

$$\frac{dR_r}{d\tau} = Er\left(\widetilde{\Theta}_{t-1}\right)^2 - R_r$$

It is straightforward to see that the variance estimators converge under the ODE and so this larger system is governed by the smaller ODE

$$\begin{bmatrix} \frac{d\widehat{\lambda}}{d\tau} \\ \frac{d\widehat{\varphi}}{d\tau} \\ \frac{d\widehat{\gamma}}{d\tau} \\ \frac{d\widehat{\chi}}{d\tau} \end{bmatrix} = \begin{bmatrix} \lambda \\ \varphi \\ \gamma \\ 1 \end{bmatrix} - I_4 \begin{bmatrix} \widehat{\lambda} \\ \widehat{\varphi} \\ \widehat{\gamma} \\ \widehat{\chi} \end{bmatrix}$$

and since the eigenvalues of this system are all -1, the ODE is stable.

The associated ODE for the private sector RLS learning are as before, but now also depends on the parameter estimates of the central bank. So local stability of the REE will depend on the stability of the smaller ODE

$$\frac{d\xi_1}{d\tau} = \begin{bmatrix} \left(\beta + \widehat{\lambda}\left(\widehat{\varphi} - \widehat{\gamma}\right)\right)\phi_1 + \widehat{\lambda}\phi_2 - \widehat{\lambda}\widehat{\gamma}\phi_3 \\ \rho\left(+\left(\beta + \widehat{\lambda}\left(\widehat{\varphi} - \widehat{\gamma}\right)\right)\psi_1 + \widehat{\lambda}\psi_2 - \widehat{\lambda}\widehat{\gamma}\psi_3 + 1 - \widehat{\lambda}\widehat{\delta}_u\left(\widehat{\varphi} + \widehat{\gamma}\right)\right) \end{bmatrix} - \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}$$

$$\frac{d\xi_2}{d\tau} = \begin{bmatrix} \phi_1\left(\widehat{\varphi} - \widehat{\gamma}\right) + \phi_2 - \widehat{\gamma}\phi_3 \\ \rho\left(\left(\widehat{\varphi} - \widehat{\gamma}\right)\psi_1 + \psi_2 - \widehat{\gamma}\psi_3 - \widehat{\delta}_u\left(\widehat{\varphi} + \widehat{\gamma}\right)\right) \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}$$

$$\frac{d\xi_3}{d\tau} = \begin{bmatrix} -\phi_1 + \phi_3 \\ \rho\left(-\psi_1 + \psi_3 + \widehat{\delta}_u\right) \end{bmatrix} - \begin{bmatrix} \phi_3 \\ \psi_3 \end{bmatrix}$$

Given the convergence results for $\left(\widehat{\lambda}_t, \widehat{\varphi}_t, \widehat{\gamma}_t, \widehat{\chi}_t\right)' \to (\lambda, \varphi, \gamma, \chi)'$ then the result from Proposition 2 holds for the stability of private sector learning as well. That is, central bank learning converges to the REE optimal fundamentals-based rule, *conditional on* the stability of private sector expectations. However, private sector learning converges to the required REE parameters with probability zero as in Proposition 2. Thus central-bank learning would not be expectationally stable either. The proof is similar for the case of SG algorithms.

D Proof of Proposition 4

Private agents are again learning using the forecast functions and learning rules (RLS or SG) shown in (16)-(26). Now, the local stability of the learning rules are governed by the ODEs:

$$\begin{bmatrix} \frac{d\phi_1}{d\tau} \\ \frac{d\phi_2}{d\tau} \\ \frac{d\phi_3}{d\tau} \end{bmatrix} = \left[B^* - I_3\right] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{d\psi_1}{d\tau} \\ \frac{d\psi_2}{d\tau} \\ \frac{d\psi_3}{d\tau} \end{bmatrix} = \rho\left[B^* - I_3\right] \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} + \rho C$$

where

$$B^{*}-I_{3}=\left[\begin{array}{ccc}\frac{\beta\theta}{\lambda^{2}+\theta}-1 & 0 & 0\\ -\frac{\lambda\beta}{\lambda^{2}+\theta} & -1 & 0\\ \frac{\lambda\beta}{\left(\lambda^{2}+\theta\right)(\varphi+\gamma)} & \frac{1}{\varphi+\gamma} & \frac{\varphi-\gamma}{\varphi+\gamma}\end{array}\right].$$

The eigenvalues of $B^{*}-I_{3}$ are $-1, \frac{\varphi-\gamma}{\varphi+\gamma}, -\frac{\theta(1-\beta)+\lambda^{2}}{\theta+\lambda^{2}}$. Clearly, all eigenvalues are negative except $\frac{\varphi-\gamma}{\varphi+\gamma}$, which is negative if and only if $|\gamma|>|\varphi|$ since $\gamma<0$. Thus $B^{*}-I_{3}$ will be a stable matrix, and therefore private sector RLS or SG learning will converge to the REE, if and only if $|\gamma|>|\varphi|$.

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