International Capital Flows[1]

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Abstract

The sharp increase in both gross and net capital flows over the past two decades has lead to a renewed interest in their determinants. Most existing theories of international capital flows are in the context of models with only one asset, implying that two-way flows are impossible and there is no role for portfolio choice. In this paper we develop a method for solving stochastic general equilibrium open-economy models of portfolio choice. We show why standard first- and second-order methods to solve stochastic general equilibrium models no longer work in the presence of portfolio choice. We then extend the standard solution method in a way that gives special treatment to the optimality conditions for portfolio choice. We apply the solution method to a particular two-country, two-good, two-asset model and show that it leads to a much richer understanding of both gross and net capital flows. The approach highlights time-varying portfolio shares as a potential key source of international capital flows. The model also illustrates the role of expected and unexpected valuation effects in the external adjustment process, which have received significant attention in recent years.

1 Introduction

The last two decades have witnessed a remarkable growth of both gross and net international capital flows and external positions. The near-tripling of gross positions among industrialized countries has also given rise to large valuation effects as asset price and exchange rate changes interact with much bigger external assets and liabilities.[1] These developments have lead to a renewed interest in understanding the driving forces behind capital flows and their macroeconomic implications. Most of what we know about capital flows is within settings where only one risk-free bond is traded. Such models imply that there are no two-way capital flows, no portfolio choice and no valuation effects. At the other extreme are models where financial markets are complete. But capital flows don't really matter in these models and are rarely ever computed as the real allocation is independent of the exact structure of asset markets.[2] A broad consensus has therefore recently developed of the need for general equilibrium models of portfolio choice in which financial markets are not restricted to be complete.[3]

The goal of this paper is twofold. First, we develop a tractable method for solving general equilibrium open-economy models of portfolio choice that can be implemented both when asset markets are complete and incomplete. Second, the method is applied to a particular two-country, two-good, two-asset model to both illustrate the solution technique and to show that it can lead to a much richer understanding of both *gross* and *net* capital flows and positions, and corresponding

[1] Lane and Milesi-Ferretti (2005) offer a detailed review of these developments.

[2] The magnitude of capital flows in complete markets models depends on the particular structure through which the market completeness is implemented. In a setup where a full set of Arrow Debreu securities covering all possible future contingencies is traded in an initial period, subsequent capital flows will be always be zero. In other asset market structures with complete markets capital flows will generally be non-zero (e.g. Kollman (2006)), but Obstfeld and Rogoff (1996) argue that then they are "...merely an accounting device for tracking the international distribution of new equity claims foreigners must buy to maintain the efficient global pooling of national output risks."

[3] Typical of current views, Gourinchas (2006) writes "Looking ahead, the next obvious step is to build general equilibrium models of international portfolio allocation with incomplete markets. I see this as a major task that will close a much needed gap in the literature...". Also emphasizing the need for incomplete market models, Obstfeld (2004) writes: "at the moment we have no integrative general-equilibrium monetary model of international portfolio choice, although we need one."

adjustments of goods and asset prices. Our approach highlights a potential key source of international capital flows, associated with changes over time in portfolio shares.[4] We show that capital flows can be broken down into a component associated with portfolio growth through savings and a component associated with the optimal reallocation of portfolios across various assets. The model also allows us to study the impact of both expected and unexpected valuation effects that have received significant attention in recent years, e.g. Gourinchas and Rey (2006), Lane and Milesi-Ferretti (2005) and Tille (2005).[5]

Standard solution methods of stochastic general equilibrium models separately analyze model equations at different orders (zero-order, first-order, and so on). The zero-order component of a variable is its steady state.[6] The first-order component of a stochastic variable is proportional to model innovations, while the second-order component depends on the product of model innovations (product of first-order variables). The standard solution method computes the zero-order component of the variables from the zero-order order component of the model equations, the first-order component of the variables from the first-order component of the model equations (after linearization), and so on.

Unfortunately the standard method cannot be applied to a model with portfolio choice. For example, the zero-order component of portfolio shares cannot be computed from the zero-order component of model equations because portfolio choice is not well-defined in a deterministic environment. More generally we show that the problem is concentrated in one dimension, namely the *difference* between portfolio shares of Home and Foreign investors (i.e. the share of one asset in the Home investor's portfolio minus the share of that asset in the Foreign investor's portfolio).[7] We show that the first-order component of portfolio share differences

[4]Even in complete market models authors generally only solve the steady portfolio allocation rather than its time variation, e.g. Engel and Matsumoto (2005), Heathcote and Perri (2005) and Kollman (2006).

[5]In the United States this has been particularly important in recent years. During 2002-2005 the United States experienced very favorable valuation effects associated with the depreciation of the dollar and strong performance of European asset markets. These favorable valuation effects allowed the U.S. net external debt to be stabilized over this period even though it experienced an enormous $2.5 trillion cumulative current account deficit.

[6]Throughout the paper we define the steady state of a variable as the level it reaches when no shock has occurred for an infinitely long time.

[7]The portfolio share difference is closely connected to the concept of portfolio home bias.

cannot be solved from the first-order component of model equations. In contrast, the first-order component of *average* portfolio shares (i.e. the average fraction invested by Home and Foreign investors in an asset) is determined by the first-order component of asset market clearing conditions.

The difficulty in solving the difference in portfolio shares is best illustrated by considering its zero-order component. The steady state portfolio allocation will depend on second-order moments such as the variance and covariance of asset returns. These second-order moments only show up in the second-order component of the optimality conditions for portfolio choice. We show that solving the zero-order component of portfolio share differences is based on the second-order component of the optimality conditions for portfolio choice. Analogously, the first-order component of portfolio share differences is based on the third-order component of the optimality conditions for portfolio choice. While the third-order component of model equations is generally considered to be very small and best ignored, we show that this is misleading as it is key to obtaining the first order solution of portfolio shares.

Overall the method can be summarized as follows. The zero-order component of portfolio share differences is solved jointly with the first-order component of other model variables. This uses the second-order component of the optimality conditions for portfolio choice and the first order component of other model equations. Taking this one step further, the first-order component of portfolio share differences is solved jointly with the second-order component of other model variables. This uses the third-order component of the optimality conditions for portfolio choice and the second-order component of other model equations.

Solving for the first-order component of portfolio share differences is technically challenging as it is based on the second and third-order components of model equations. However, we show that this is only needed to solve *gross* capital flows and *gross* external assets and liabilities. It is not needed to solve for the first-order component of *net* capital flows and the *net* external asset position. Those are computed using the much easier first step of the solution method, combining the second-order component of the optimality conditions for portfolio choice with the first-order component of other model equations. This gives us the first-order component of average portfolio shares, which affects net capital flows through a reallocation towards Home assets by the average investor.

The two papers that are most closely related to ours are Devereux and Sutherland (2006a) and Evans and Hnatkovska (2005). Both papers consider general equilibrium models with portfolio choice under incomplete markets. Devereux and Sutherland (2006a) are the first to show how one can derive a solution for the zero-order (steady state) component of portfolio allocation in a broad class of such models. They combine the second-order component of optimality conditions for portfolio choice with the first-order component of other model equations. Their work is important as gross positions can have substantial macroeconomic implications. They however do not analyze the time-variation in portfolio shares, nor the dynamics in gross and net capital flows.[8] Evans and Hnatkovska (2005) solve for the dynamics of the portfolio allocation in a discrete time model by combining discrete time and continuous time solution techniques. Our approach differs from theirs in that we solve the model entirely in a discrete time setup.[9] This also has the advantage that it is more closely linked to existing solution methods of discrete time stochastic general equilibrium models. In addition we show that a large portion of our results, such as the dynamics of net asset positions and net capital flows, can be solved while keeping the technical complexity to a minimum.

The remainder of the paper is organized as follows. In section 2 we describe the solution method in general terms. Section 3 describes a particular model to which the solution method is applied. Section 4 describes the solution method in the context of the model. For a particular parameterization, section 5 discusses the implications of the model for gross and net capital flows and positions, as well as asset prices and the real exchange rate. Section 6 concludes.

[8]Building on Devereux and Sutherland (2006a), Devereux and Sutherland (2006b) consider the impact of monetary policy rules in such models. They also discuss the instantaneous impact of productivity and monetary shocks on the current account, but do not consider the dynamic response of net capital flows and the role of portfolio reallocation. Another related paper is Devereux and Saito (2005), who present an elegant analytical solution for incomplete markets model in a continuous-time setting. The analytical solution is however only possible in special cases.

[9]An additional advantage is that our solution method is not constrained to logarithmic preferences.

2 A general description of the solution method

2.1 Overview

This section describes the key features of our approach. We start by presenting the breakdown of the model equations and variables into components of different orders. We then illustrate how these components are used in standard first- and second-order solution methods for models that abstract from portfolio choice. The next step shows why this standard approach no longer works in a model with portfolio choice. The section ends by describing how the method is adapted to encompass portfolio choice and discusses a solution algorithm for solving general equilibrium models.

2.2 The various orders of approximation

Dynamic general equilibrium models generally lead to a set of equations of the form:

$$E_t f(x_t, x_{t+1}) = 0 \tag{1}$$

where x_t contains a vector of both control and state variables at time t. The number of equations, Z, is equal to the number of variables in x_t. A subset of the state variables, denoted y_t, such as productivity processes, usually follows an exogenous process:

$$y_{t+1} = \Lambda y_t + \epsilon_{t+1} \tag{2}$$

where ϵ_{t+1} are the model innovations. Each variable has components that are zero-order, first-order, and higher order:

$$x_t = x(0) + x_t(1) + x_t(2) + ... \tag{3}$$

$x(0)$ is the zero-order component of x_t, which is the steady state value of x_t around which the model equations are expanded. $x_t(O)$ is the order O component, for $O > 0$. For instance, $x_t(2)$ is the second-order component of x_t. Normalizing the standard deviation of all model innovations to σ, the order of a variable is defined as follows:

Definition 1 *The component of a variable is of order O if:*

$$\lim_{\sigma \to 0} \frac{x_t(O)}{\sigma^O}$$

is either a well-defined stochastic variable whose variance does not depend on σ or a non-zero constant that does not depend on σ.

Components of order O are proportional to σ^O. Zero-order variables are non-zero constant terms that do not depend on σ such as the steady state $p(0)$ of goods prices. Stochastic variables that are proportional to model innovations are first-order. An example is the dynamics of goods prices in response to a shock: $p_{t+1}(1) = p_1 \epsilon_{t+1}$. Stochastic variables that depend on the product of model innovations are second-order, such as $p_{t+1}(2) = p_2 (\epsilon_{t+1})^2$. Examples of third-order variables are the product of three model innovations, or the product of σ^2 and a model innovation.

(1) can be written as an infinite order Taylor expansion around the allocation $x_t = x_{t+1} = x(0)$. Let f_1 and f_2 denote the first-order derivatives of f with respect to respectively x_t and x_{t+1}, both evaluated at $x(0)$. Second-order derivatives f_{11}, f_{22} and f_{12} are defined analogously. Writing $\hat{x}_t = x_t - x(0)$, and limiting ourselves for illustrative purposes to a second-order expansion, we have:

$$f(x_t, x_{t+1}) = f(x(0), x(0)) + f_1 \hat{x}_t + f_2 \hat{x}_{t+1} + \frac{1}{2} \hat{x}_t' f_{11} \hat{x}_t + \frac{1}{2} \hat{x}_{t+1}' f_{22} \hat{x}_{t+1} + \hat{x}_t' f_{12} \hat{x}_{t+1} + ...$$

Substituting $\hat{x}_t = x_t(1) + x_t(2) + ...$ in this relation and taking expectations, we write the zero-order component of (1) as

$$f(x(0), x(0)) = 0 \tag{4}$$

Similarly, the first-order component is

$$f_1 x_t(1) + f_2 E_t x_{t+1}(1) = 0 \tag{5}$$

which consists only of linear terms. The second-order component is

$$\begin{aligned} 0 = {} & f_1 x_t(2) + f_2 E_t x_{t+1}(2) + \frac{1}{2} x_t'(1) f_{11} x_t(1) \\ & + \frac{1}{2} E_t x_{t+1}'(1) f_{22} x_{t+1}(1) + E_t x_t'(1) f_{12} x_{t+1}(1) \end{aligned} \tag{6}$$

Notice that the second-order component includes linear terms. Therefore, while first-order components are linear, linear terms are not necessarily made only of first-order components.

2.3 Standard solution without portfolio choice

The standard method for solving dynamic stochastic general equilibrium models solves the order O component of model variables from the order O component of model equations, with O usually limited to 1 or 2.

The solution is sequential in that the solution of the order O component of model variables requires the solution at lower orders. The zero-order component of variables is obtained from the deterministic steady state (4). The first-order dynamics is then obtained using a linear approximation of the model equations around the steady state, (2) and (5). The terms f_1 and f_2 in (5) are computed using the zero-order solution. Finally, the second-order component of model variables is solved from (6). f_1, f_2, f_{11}, f_{22} and f_{12} in (6) are computed using the zero-order solution, while $x_t(1)$ and $x_{t+1}(1)$ use the first-order solution.

2.4 Introducing portfolio choice

When solving a model with portfolio choice the standard solution method unfortunately no longer works. But before showing this, it is useful to specify how portfolio shares enter the model.

Assumption 1 *The only two ways that portfolio shares enter model equations are (i) through the return on the overall portfolio and (ii) through asset demand.*

This assumption is not very limiting and holds in almost any general equilibrium model with portfolio choice. Portfolio shares clearly affect the overall portfolio return, which determines wealth accumulation. For concreteness, assume that there are two countries, Home and Foreign, and N assets with asset i providing a gross stochastic return $R_{i,t+1}$ from t to $t+1$, with the return expressed in units of a numeraire currency. Consider an investor in the Home country. In period t she invests a share $k_{i,t}^H$ of her wealth in asset i, with the shares summing up to 1. Treating asset 1 as a base asset, we can write the portfolio return as

$$R_{t+1}^{p,H} = \sum_{i=1}^{N} k_{i,t}^H R_{i,t+1} = R_{1,t+1} + \sum_{i=2}^{N} k_{i,t}^H ER_{i,t+1}$$

where $ER_{i,t+1} = R_{i,t+1} - R_{1,t+1}$ is the excess return on asset i, relative to the base asset.

The second way portfolio shares are assumed to affect the model is through asset demand. This shows up in the asset market clearing condition for asset i:

$$Q_{i,t}K_{i,t} = k^H_{i,t}W_t + k^F_{i,t+1}W^*_t \tag{7}$$

The left hand side of (7) is the value of the asset supply, which is the product of the asset price $Q_{i,t}$ and the quantity of the asset available for trading, $K_{i,t}$. The right hand side of (7) is the asset demand from both Home and Foreign investors. The Home investor invests a share $k^H_{i,t}$ of her wealth W_t in asset i, and the Foreign investor invests a share $k^F_{i,t}$ of her wealth W^*_t in the asset.

By affecting portfolio returns, which affect future wealth and consumption, portfolio shares also affect asset pricing kernels. We will write the asset pricing kernels for the Home and Foreign investors as $m^H(x_t, x_{t+1})$ and $m^F(x_t, x_{t+1})$. The optimality conditions for portfolio choice are then:

$$E_t m^H(x_t, x_{t+1})ER_{i,t+1} = 0 \qquad i = 2, .., N \tag{8}$$
$$E_t m^F(x_t, x_{t+1})ER_{i,t+1} = 0 \qquad i = 2, .., N \tag{9}$$

These show that investors choose their portfolio to equalize the expected return on each asset, discounted by the pricing kernel. An immediate implication of (8)-(9) is that the zero-order components of assets returns are the same: $R_i(0) = R(0)$ for any $i = 1, .., N$.

Rather than conducting the analysis in terms of the portfolio shares of each country, it is useful to do so in terms of *average* portfolio shares and *differences* in portfolio shares. The average portfolio share of asset i is the average of its share in the portfolios of the Home and Foreign investors. It measures the extent to which investors worldwide hold asset i:

$$k^A_{i,t} = \frac{1}{2}\left(k^H_{i,t} + k^F_{i,t}\right) \tag{10}$$

The difference in portfolio shares of asset i measures the extent to which Home investors hold more or less of the asset relative to Foreign investors:

$$k^D_{i,t} = k^H_{i,t} - k^F_{i,t} \tag{11}$$

$k^D_{i,t} > 0$ indicates that the share of asset i in a Home investor's portfolio exceeds the share in a Foreign investor's portfolio. If asset i is equity in Home firms, this is the well-known *home bias* in portfolios. The shares in each portfolio are simple combinations of (10) and (11): $k_{i,t}^H = 0.5k_{i,t}^D + k_{i,t}^A$ and $k_{i,t}^F = -0.5k_{i,t}^D + k_{i,t}^A$.

We define similar measures of average wealth and cross-country differences:

$$W_t^A = \frac{1}{2}(W_t + W_t^*) \qquad\qquad W_t^D = W_t - W_t^*$$

Although this is not key to the argument, we assume that the zero-order components of the wealth of the two countries are the same, equal to $W(0)$.[10]

Assumption 1 has an important implication that will be key to the solution method.

Corollary 1 *The order O components of portfolio share differences $k_{i,t}^D$ do not affect the order O component of model equations for any $O \geq 0$.*

In order to see this, the order O components of the Home portfolio return and total asset demand are:

$$R_{t+1}^{p,H}(O) = R_{1,t+1}(O) + \sum_{i=2}^{N}\sum_{o=0}^{O}\left(0.5k_{i,t}^D(o) + k_{i,t}^A(o)\right) ER_{i,t+1}(O-o) \quad (12)$$

$$\sum_{o=0}^{O}\left(0.5k_{i,t}^D(o)W_t^D(O-o) + 2k_{i,t}^A(o)W_t^A(O-o)\right) \quad (13)$$

The order O component of the average portfolio share, $k_{i,t}^A(O)$, enters (13) and can therefore be identified from the order O component of the asset market clearing equations. By contrast, the order O component of the difference in portfolio shares, $k_{i,t}^D(O)$, does not enter either (12) or (13), and we therefore cannot compute it from the order O equations of the model. Specifically, $k_{i,t}^D(O)$ appears in (12) only multiplied with the zero-order component of excess return, $ER_{i,t+1}(0)$, which is zero. Similarly, $k_{i,t}^D(O)$ appears in (13) multiplied with the zero-order component of the wealth difference, $W_t^D(0)$, which is also zero.

While the order O component of $k_{i,t}^D$ does not affect the order O component of model equations, it is clear that the $O-1$ and lower order components of $k_{i,t}^D$ do affect the order O component of model equations (they affect both (12) and (13)). This will be key to the solution method discussed below.

[10] Otherwise average portfolio shares need to be defined as a weighted average, using the zero-order components of wealth shares as weights.

2.5 The limitation of the standard solution method

The standard method of solving the order O component of model variables from the order O component of model equations only works if the following two necessary conditions are satisfied:

Condition 1 *The order O components of all model equations depend on the order O component of at least one model variable.*

Condition 2 *The order O components of all model variables affect the order O component of at least one model equation.*

These conditions are needed to solve the order O components of variables from the order O components of equations.

Neither of these conditions holds in the presence of portfolio choice. First, Corollary 1 implies that Condition 2 is not satisfied. The order O components of the $N-1$ portfolio share differences, $k^D_{i,t}$, do not affect the order O components of any model equation, as discussed above. Second, Condition 1 is also not satisfied because there are $N-1$ equations whose order O components do not depend on order O variables.

To see this, consider the order O component of the optimality conditions for portfolio choice of Home and Foreign investors:[11]

$$E_t ER_{i,t+1}(O) + E_t \sum_{o=1}^{O} m^H_{t+1}(o) ER_{i,t+1}(O-o) = 0 \quad i = 2,..,N \tag{14}$$

$$E_t ER_{i,t+1}(O) + E_t \sum_{o=1}^{O} m^F_{t+1}(o) ER_{i,t+1}(O-o) = 0 \quad i = 2,..,N \tag{15}$$

(14)-(15) show that the order O component of the expected excess return depends on the order O component of the covariance between the asset pricing kernel and the excess return.[12] The order O component of the asset pricing kernels does not enter (14)-(15) since it is multiplied by zero-order components of the excess returns, $ER_{i,t+1}(0)$, which are zero. (14)-(15) imply that for both Home and Foreign investors the first-order component of expected excess returns is zero: $E_t ER_{i,t+1}(1) = 0$.

[11] Without loss of generality the zero-order components of the asset pricing kernels are normalized at 1.

[12] A covariance is often considered to be a second-order moment, but it can have higher order components as well when it is not constant over time.

Now consider the difference between the Home and Foreign optimality conditions for portfolio choice (14) and (15), which we will refer to as the portfolio Euler equation differentials. The zero and first-order components of the difference are zero. For $O \geq 2$ the difference is

$$E_t \sum_{o=1}^{O} \left[m_{t+1}^H(o) - m_{t+1}^F(o) \right] ER_{i,t+1}(O-o) = 0 \qquad i = 2, .., N \tag{16}$$

While this difference is not zero, it still does not depend on the order O component of variables because $ER_{i,t+1}(0) = 0$. There are therefore $N-1$ equations for which the order O component does not depend on the order O component of any model variable. Clearly therefore, both Conditions 1 and 2 are not satisfied.[13]

Two final comments are worth making with regards to (16), which will be relevant to the solution algorithm discussed below. First, while the order O components of the portfolio Euler equation differentials do not depend on the order O components of model variables, they do depend on the order $O-1$ components of model variables, as reflected in both $m_{t+1}^H(O-1) - m_{t+1}^F(O-1)$ and $ER_{i,t+1}(O-1)$. Second, portfolio share differences do not directly affect (16). They only impact the asset pricing kernels indirectly through the portfolio return, which affects next period's wealth. To summarize, the order O components of portfolio Euler equation differentials depend on components of order $O-1$ and less of model variables other than portfolio share differences, which enter only indirectly through the portfolio return.

2.6 Solution algorithm

In developing the solution method, we start from the fact that Conditions 1 and 2 are satisfied for $\tilde{Z} = Z - (N-1)$ equations and variables. This includes all model variables other than the vector k_t^D of $N-1$ portfolio share differences and all model equations other than the $N-1$ portfolio Euler equation differentials (16). From now on we will simply refer to these as the "other" model variables and "other" model equations. This suggests that the order O components of the

[13] (16) is derived under the assumption that the return on asset i is the same for Home and Foreign investors, in terms of the numeraire. The model presented in Section 3 relaxes this assumption by introducing a trading friction, which appears as an additional term in (16). But the presence of this additional term does not affect our point that the order O component of (16) does not depend on the order O component of model variables.

$\tilde{Z}$ "other" model variables can be solved from the order O components of the $\tilde{Z}$ "other" model equations using the standard method. However, this solution is conditional on $k_t^D(O-1)$ as we have shown that the order $O-1$ components of portfolio share differences affect the order O components of the "other" model equations.

In order to simultaneously solve for $k_t^D(O-1)$ we need to use the order $O+1$ components of the $N-1$ portfolio Euler equation differentials (16). The latter depend on the order O components of the "other" model variables, which in turn are solved as a function of $k_t^D(O-1)$ from the "other" model equations. We can therefore use (16) to solve for $k_t^D(O-1)$. This suggests a joint solution of $k_t^D(O-1)$ and the order O components of "other" model variables. The solution algorithm can be summarized as follows.

Solution Algorithm *In sequence $O = 1, 2, ..$ solve the order $O-1$ component of k_t^D jointly with the order O components of all "other" model variables, using (i) the order $O+1$ components of the portfolio Euler equation differentials and (ii) the order O components of all "other" model equations.*

Consider the case of $O = 1$. We know from (12)-(13) that the first-order component of model equations is affected by the zero-order component of k_t^D, namely $k_t^D(0)$, but not the first-order component of k_t^D, that is $k_t^D(1)$. Using the first-order component of the $\tilde{Z}$ "other" model equations, we can then solve the first-order component of the $\tilde{Z}$ "other" variables as a function of the unknown $k_t^D(0)$. To solve for $k_t^D(0)$, we then use the second-order component of the portfolio Euler equation differentials. These depend on the first-order components of the "other" model variables, which have been solved as a function of $k_t^D(0)$. Therefore $k_t^D(0)$ and the first-order components of the "other" model variables are solved jointly, using the first-order components of "other" model equations and the second-order component of (16).[14]

For $O = 2$ we follow the same steps one order higher, jointly solving the first-order component of k_t^D and the second-order component of the $\tilde{Z}$ "other" model variables. In this case we use the second-order components of the "other" model equations together with the third-order component of the portfolio Euler equation differentials. This is where we stop in the paper as we are only interested in the

[14] This method to jointly solve $k_t^D(0)$ and the first-order components of the "other" variables corresponds to the method of Devereux and Sutherland (2006a).

first-order components of gross and net capital flows. But in principle one can keep following this algorithm for higher orders.

Solving for the first-order component of k_t^D requires second and third-order components of model equations and is therefore substantially more complicated than solving the first-order component of "other" model variables. However, the first-order solution of k_t^D is only needed to compute the first-order component of *gross* capital flows and *gross* external positions. We can gain substantial insights on the solution of the model, while avoiding technical complication, if we are willing to focus on the *net* asset positions and *net* capital flows. The first-order components of these net variables do not depend on the first-order solution of k_t^D. Solving the zero-order solution of k_t^D jointly with the first-order solution of the other model variables is sufficient.

This can be seen as follows. If the first J assets are claims on the Home country, the net value of Home external assets minus liabilities is

$$W_t \sum_{i=J+1}^{N} k_{i,t}^H - W_t^* \sum_{i=1}^{J} k_{i,t}^F = W_t - W_t \sum_{i=1}^{J} k_{i,t}^H - W_t^* \sum_{i=1}^{J} k_{i,t}^F$$

The first-order component of the net external asset position is proportional to:

$$W_t(1) - 2W_t^A(1) \sum_{i=1}^{J} k_{i,t}^A(0) - 2W(0) \sum_{i=1}^{J} k_{i,t}^A(1) - \frac{1}{2} W_t^D(1) \sum_{i=1}^{J} k_{i,t}^D(0)$$

It clearly depends on the zero and first-order components of the average portfolio shares, but only on the zero-order component of the difference in portfolio shares. Net capital flows are simply equal to the change in the net external asset position, after controlling for valuation changes associated with asset prices, and can also be solved without needing the first-order difference in portfolio shares.

3 A two-country, two-good, two-asset model

This section describes a symmetric two-country, two-good, two-asset model to which the solution technique will be applied. In order to both simplify and focus the analysis, the key element of the model is portfolio choice. We abstract from consumption and investment decisions, even though they can easily be introduced in extensions. By focusing squarely on portfolio decisions, the model aims directly at what has been the key obstacle so far in solving incomplete markets models.

3.1 Two goods: production and consumption

The two countries, Home and Foreign, each produce a different good that is available for consumption in both countries. Production uses a constant returns to scale technology combining labor and capital:

$$Y_{H,t} = A_{H,t} K_{H,t}^{1-\theta} N_{H,t}^{\theta} \qquad\qquad Y_{F,t} = A_{F,t} K_{F,t}^{1-\theta} N_{F,t}^{\theta}$$

where Y_H and Y_F denote outputs of the Home and Foreign goods respectively, A is an exogenous stochastic productivity term, K is the capital input and N the labor input. A share θ of output is paid to labor, with the remaining going to capital. The capital stocks and labor inputs are exogenous and normalized to unity, so outputs simply reflect the levels of productivity:

$$Y_{H,t} = A_{H,t} \qquad\qquad Y_{F,t} = A_{F,t} \tag{17}$$

Log-productivity in both countries is assumed to follow an exogenous autoregressive process:

$$a_{H,t+1} = \rho a_{H,t} + \epsilon_{H,t+1} \qquad\qquad a_{F,t+1} = \rho a_{F,t} + \epsilon_{F,t+1} \tag{18}$$

where lower case letters denote logs and $\rho \in (0,1)$. The innovations have a $N(0,\sigma^2)$ distribution and are uncorrelated across countries.

Both goods are used for consumption, with the overall consumption index in both countries reflecting a preference towards domestic goods:

$$\begin{aligned} C_t &= \left[(\alpha)^{\frac{1}{\lambda}} (C_{H,t})^{\frac{\lambda-1}{\lambda}} + (1-\alpha)^{\frac{1}{\lambda}} (C_{F,t})^{\frac{\lambda-1}{\lambda}} \right]^{\frac{\lambda}{\lambda-1}} \\ C_t^* &= \left[(1-\alpha)^{\frac{1}{\lambda}} \left(C_{H,t}^*\right)^{\frac{\lambda-1}{\lambda}} + (\alpha)^{\frac{1}{\lambda}} \left(C_{F,t}^*\right)^{\frac{\lambda-1}{\lambda}} \right]^{\frac{\lambda}{\lambda-1}} \end{aligned}$$

C is the overall consumption index of the Home consumer, C_H denotes her consumption of Home goods and C_F denotes her consumption of Foreign goods. The index for the Foreign consumer is similar, with an asterisk superscript added. λ is the elasticity of substitution between Home and Foreign goods. α is relative preference towards domestic goods, with $\alpha > 0.5$ corresponding to home bias in consumption.

The allocation of consumption across goods is computed along the usual lines. For instance, the demands for Foreign goods by Home and Foreign consumers are:

$$C_{F,t} = (1-\alpha) (P_{F,t})^{-\lambda} (P_t)^{\lambda} C_t \qquad\qquad C_{F,t}^* = \alpha (P_{F,t})^{-\lambda} (P_t^*)^{\lambda} C_t^* \tag{19}$$

The Home good is the numeraire. $P_{F,t}$ is the relative price of the Foreign good and P_t and P_t^* are the consumer price indices:

$$P_t = \left[\alpha + (1-\alpha)\left[P_{F,t}\right]^{1-\lambda}\right]^{\frac{1}{1-\lambda}} \qquad P_t^* = \left[(1-\alpha) + \alpha\left[P_{F,t}\right]^{1-\lambda}\right]^{\frac{1}{1-\lambda}}$$

The presence of home bias in consumption, $\alpha > 0.5$, implies that the Home and Foreign consumer price indexes do not move in step, so movements in the relative price of the Foreign good lead to movements in the real exchange rate P_t^*/P_t. The model therefore has implications for real exchange rate adjustments that can be expected when faced with external imbalances, as in Obstfeld and Rogoff (2005) and Engel and Rogers (2006).

3.2 Two assets: rates of return

Two assets are traded, claims on the Home capital stock K_H and claims on the Foreign capital stock K_F. We refer to these assets as Home equity and Foreign equity. The price at time t of a unit of Home equity is denoted by $Q_{H,t}$, measured in terms of the numeraire Home good. The holder of this claim gets the dividend at period $t+1$, which is a share $1-\theta$ of output (17), and can sell the claim for a price $Q_{H,t+1}$. The overall return on a Home equity, in terms of Home goods, is then:

$$R_{H,t+1} = 1 + \frac{Q_{H,t+1} - Q_{H,t}}{Q_{H,t}} + \frac{(1-\theta)A_{H,t+1}}{Q_{H,t}} \tag{20}$$

Similarly, the price at time t of a unit of Foreign equity is denoted by $Q_{F,t}$, expressed in terms of the numeraire Home good. The return on Foreign equity is:

$$R_{F,t+1} = 1 + \frac{Q_{F,t+1} - Q_{F,t}}{Q_{F,t}} + \frac{(1-\theta)P_{F,t+1}A_{F,t+1}}{Q_{F,t}} \tag{21}$$

(20)-(21) show that the returns consist of a capital gain or loss due to movements in equity prices and a dividend yield.

While agents can invest in equity abroad, this entails a cost. Specifically, the agent receives only the returns in (20)-(21) times an iceberg cost $\exp\left[-\tau^2\right] \in (0,1)$. This cost does not represent lost resources, but instead is a fee paid to a broker. It is a simple way to capture the hurdles of investing outside the domestic country, reflecting the cost of gathering information on an unfamiliar market for instance. The cost is second-order (τ^2 is proportional to σ^2) to ensure a well-behaved portfolio allocation.

In period t a Home agent invests a fraction $k_{H,t}^{H}$ of her wealth in Home equity, and a fraction $k_{F,t}^{H} = 1 - k_{H,t}^{H}$ in Foreign equity. The overall real return on her portfolio, expressed in terms of the Home consumption basket, is then:

$$R_{t+1}^{p,H} = \left[k_{H,t}^{H} R_{H,t+1} + (1 - k_{H,t}^{H}) e^{-\tau^2} R_{F,t+1}\right] \frac{P_t}{P_{t+1}} \tag{22}$$

Similarly, a Foreign agent invests a fraction $k_{H,t}^{F}$ of her wealth in Home equity, and a fraction $k_{F,t}^{F} = 1 - k_{H,t}^{F}$ in Foreign equity, leading to an overall real return in terms of the Foreign consumption basket of:

$$R_{t+1}^{p,F} = \left[k_{H,t}^{F} e^{-\tau^2} R_{H,t+1} + (1 - k_{H,t}^{F}) R_{F,t+1}\right] \frac{P_t^*}{P_{t+1}^*} \tag{23}$$

3.3 Wealth accumulation

It is well-known that when financial markets are incomplete even transitory shocks can lead to a non-stationary world distribution of wealth, so that the steady state is not well-defined. In order to induce stationarity, and also abstract from consumption-savings decisions, we adopt the framework of Caballero, Fahri and Gourinchas (2006). A fraction ψ of agents dies each period. An identical number of agents is born, keeping the population constant. Agents only consume in the last period of life, during which they liquidate all their assets. Since the probability of death is the same for all agents, total consumption is then simply equal to aggregate wealth times the probability of death.

As an additional simplification we assume that newborn agents work only in the first period of their life and therefore face no risk on any future labor income. After that the wealth of a particular Home investor j accumulates according to

$$W_{t+1}^{j} = W_t^{j} R_{t+1}^{p,H} \tag{24}$$

The portfolio return will be the same for all Home investors as they all choose the same portfolio.

Aggregate wealth accumulation is different for three reasons. First, only a fraction $1 - \psi$ of wealth is reinvested since the rest is consumed by agents who will die. Second, labor income of the newborn raises aggregate wealth. Third, we assume that the cost of equity investment abroad does not represent lost resources, but instead is a fee paid to a broker. For simplicity we assume that the brokers are the newborn agents. These fees therefore redistribute wealth between the newborn and older agents, but do not affect aggregate wealth. Let W_t and W_t^* be real aggregate wealth of the Home and Foreign countries, measured in terms of their respective consumption baskets. They then accumulate according to

$$W_{t+1} = (1-\psi)\left[k_{H,t}^{H} R_{H,t+1} + (1-k_{H,t}^{H}) R_{F,t+1}\right] \frac{P_t}{P_{t+1}} W_t + \frac{\theta A_{H,t+1}}{P_{t+1}} \tag{25}$$

$$W_{t+1}^* = (1-\psi)\left[k_{H,t}^{F} R_{H,t+1} + (1-k_{H,t}^{F}) R_{F,t+1}\right] \frac{P_t^*}{P_{t+1}^*} W_t^* + \frac{\theta P_{F,t+1} A_{F,t+1}}{P_{t+1}^*} \tag{26}$$

3.4 Markets clearing

There are goods and asset market clearing conditions for both countries. Consumption by the Home and Foreign dying agents has to equal the output of Home and Foreign goods. Using (17) and (19), the Home and Foreign goods market clearing conditions are

$$A_{H,t} = \alpha \left(P_t\right)^{\lambda} \psi W_t + (1-\alpha)\left(P_t^*\right)^{\lambda} \psi W_t^* \tag{27}$$

$$A_{F,t} = (1-\alpha)\left(P_{F,t}\right)^{-\lambda}\left(P_t\right)^{\lambda} \psi W_t + \alpha \left(P_{F,t}\right)^{-\lambda}\left(P_t^*\right)^{\lambda} \psi W_t^* \tag{28}$$

Turning to asset markets, the total values of Home and Foreign equity supply are equal to $Q_{H,t}$ and $Q_{F,t}$ since the capital stocks are normalized to 1. The amounts invested by Home and Foreign agents at the end of period t, measured in Home goods, are $(1-\psi) W_t P_t$ and $(1-\psi) W_t^* P_t^*$ respectively. The market clearing conditions for Home and Foreign asset markets are then

$$Q_{H,t} = (1-\psi)\left[k_{H,t}^{H} W_t P_t + k_{H,t}^{F} W_t^* P_t^*\right] \tag{29}$$

$$Q_{F,t} = (1-\psi)\left[k_{F,t}^{H} W_t P_t + k_{F,t}^{F} W_t^* P_t^*\right] \tag{30}$$

3.5 Portfolio allocation

The only decision faced by agents is the allocation of their investment between Home and Foreign equity. A Home agent j who dies in period $t+1$ consumes her entire wealth and gets a utility

$$U_{t+1}^{j} = \frac{\left(W_{t+1}^{j}\right)^{1-\gamma}}{1-\gamma}$$

From the point of view of period t, the agent faces a probability ψ of dying the next period. We denote the value of wealth in period t by $V(W_t^j)$. The Bellman equation is

$$V(W_t^j) = \beta(1-\psi)E_t V(W_{t+1}^j) + \beta\psi \frac{E_t \left(W_{t+1}^j\right)^{1-\gamma}}{1-\gamma} \tag{31}$$

where β is the discount rate. We conjecture the following form for the value of wealth:

$$V(W_t^j) = e^{v+f_H(S_t)} \frac{\left(W_t^j\right)^{1-\gamma}}{1-\gamma} \tag{32}$$

where S_t is the state space discussed below. The function $f_H(S_t)$ captures time variation in expected portfolio returns. The steady state of S_t is a vector of zeros and we normalize $f_H(0) = 0$. The constant term v can have components of zero, first and higher order, written as $v = v(0) + v(1) + ...$, with $v(i)$ proportional to σ^i. For Foreign investors the function $f_H(S_t)$ is replaced by $f_F(S_t)$.

Agent j chooses the portfolio allocation to maximize the right hand side of (31). After substituting the wealth accumulation equation (24) for agent j and using the portfolio return (22), the first-order condition with respect to the portfolio share invested in Home equity is

$$E_t \left((1-\psi)e^{v+f_H(S_{t+1})} + \psi\right) \left(R_{t+1}^{p,H}\right)^{-\gamma} \left(R_{H,t+1} - e^{-\tau^2} R_{F,t+1}\right) \frac{P_t}{P_{t+1}} = 0 \tag{33}$$

Similarly, the first-order condition for Foreign investors is

$$E_t \left((1-\psi)e^{v+f_F(S_{t+1})} + \psi\right) \left(R_{t+1}^{p,F}\right)^{-\gamma} \left(e^{-\tau^2} R_{H,t+1} - R_{F,t+1}\right) \frac{P_t^*}{P_{t+1}^*} = 0 \tag{34}$$

Using (32), the Bellman equation can be written as

$$e^{v+f_H(S_t)} = \beta E_t \left((1-\psi)e^{v+f_H(S_{t+1})} + \psi\right) \left(R_{t+1}^{p,H}\right)^{1-\gamma} \tag{35}$$

This gives an implicit solution to the function $f_H(S_t)$. The Bellman equation for Foreign investors it is

$$e^{v+f_F(S_t)} = \beta E_t \left((1-\psi)e^{v+f_F(S_{t+1})} + \psi\right) \left(R_{t+1}^{p,F}\right)^{1-\gamma} \tag{36}$$

4 Solution of the model

We now apply the general solution method discussed in section 2 to the specific model of section 3. After substitution of the expressions for asset and portfolio returns, the model can be summarized by 12 equations: the two processes for technology (18), the two wealth accumulation equations (25)-(26), the two goods market equilibrium equations (27)-(28), the two asset market clearing conditions (29)-(30), the two Euler equations for portfolio choice (33)-(34) and the two Bellman equations (35)-(36). The Foreign goods market equilibrium condition (28) can be dropped due to Walras' law, which gives a total of 11 equations.

Dropping country subscripts due to symmetry, the steady state or zero-order components of variables are $W(0) = 1/\psi$, $R(0) = (1 - \psi\theta) / (1 - \psi) > 1$, $Q(0) = (1 - \psi) / \psi$, $A(0) = P_F(0) = 1$ and $v(0) = ln(\beta\psi) + (1-\gamma)ln(R(0)) - ln(1-\beta(1-\psi))$. These follow directly from the zero-order components of all equations. As discussed in section 2, the zero-order component of portfolio shares can only be computed from a higher (second) order component of portfolio Euler equations. We write the steady state fraction that each country invests in domestic assets as $k(0)$. Therefore $k_H^H(0) = k_F^F(0) = k(0)$. To compute the higher order components of all equations we expand around the zero-order components of all variables. As standard, for variables other than portfolio shares the expansions will be around their logarithmic form, which we denote with lower case letters. Appendix A lists the model equations with variables in logarithmic form.

We can now follow the solution method described in section 2. We keep the description of the solution method as non-technical as possible, focusing on the methodology rather than the details. Appendices B and C provide an abbreviated version of technical details associated with the Bellman equations and the Euler equations for portfolio choice, with a full description of all the algebra left to a Technical Appendix that is available on request.

4.1 The easy part

We start with the first-order solution of all variables other than the portfolio share difference $k_t^D = k_{H,t}^H - k_{H,t}^F$, conditional on the zero order component $k(0)$ of portfolio shares. For technology, wealth and portfolio shares we use the differences and averages of the variables across countries rather than the country-specific variables themselves. For example, $a_t^D = a_t^H - a_t^F$ and $a_t^A = 0.5(a_t^H + a_t^F)$.

The vector of state variables is

$$S_t = \begin{pmatrix} a_t^D \\ w_t^D \\ a_t^A \end{pmatrix} \tag{37}$$

The state consist of the average and difference in technology variables and the difference in wealth levels. The average wealth level is not a separate state variable since it is tied to the average technology level through the goods market equilibrium conditions. The first-order components of w_t^A and a_t^A turn out to be identical.

First consider the 9 equations of the model other than the Bellman equations. After linearization we obtain the first-order components of the equations. There is one redundancy since the first-order component of the Euler equations for Home and Foreign investors are identical. They both imply that the first-order components of expected returns are equal:

$$E_t r_{H,t+1}(1) = E_t r_{F,t+1}(1)$$

This leaves us with 8 equations. Taking expectations of all equations, they take the form $E_t f(x_t, x_{t+1}) = 0$, where x_t consists of the 3 state variables in (37) plus the 5 control variables $cv_t = (w_t^A, p_{F,t}, k_t^A, q_{H,t}, q_{F,t})'$.

Using the entirely standard first-order solution technique applied to the first-order components of the log-linearized equations (see the Technical Appendix for details), we can then solve for the first-order component of control variables as a function of state variables and the dynamic process of the first-order component of state variables:

$$cv_t(1) = BS_t(1) \tag{38}$$

$$S_{t+1}(1) = N_1 S_t(1) + N_2 \epsilon_{t+1} \tag{39}$$

where B, N_1 and N_2 are matrices and $\epsilon_{t+1} = (\epsilon_{t+1}^H, \epsilon_{t+1}^F)'$ is the vector of technology shocks.

The first-order component of k_t^A, the average fraction invested in Home assets, only shows up in the first-order component of the asset market clearing conditions. A higher average portfolio share implies a higher demand for Home equity, which raises the relative price of Home equity. This lowers the expected return on Home equity relative to Foreign equity. Imposing that the first-order components of expected returns must be equal then identifies the equilibrium average portfolio share. As discussed in section 2, the first-order solution of the average portfolio share, not the portfolio share difference, is sufficient to compute the first-order component of the net external asset position and net capital flows.

The final two equations are the Bellman equations. Let the first-order components of $f_H(S_t)$ be $H_{1,H}S_t(1)$, where $H_{1,H}$ is the first-order derivative of f_H with respect to S_t at its steady state of $S_t(0) = (0,0,0)'$. Appendix B shows that $H_{1,H}$ can be computed from the first-order component of the Home Bellman equation, which also gives $v(1) = 0$. For the Foreign country the first-order component of $f_F(S_t)$ is $H_{1,F}S_t(1)$, with $H_{1,F}$ solved analogously from the first-order component of the Foreign Bellman equation.

4.2 A bit more difficult

The first-order solution so far is conditional on the unknown zero-order component of the portfolio share, $k(0)$. As discussed in section 2, it can be solved from the difference across countries of the second-order component of the portfolio Euler equations. Leaving the algebraic details to the Technical Appendix, we get

$$\begin{aligned} k(0) &= \frac{1}{2} + \frac{\tau^2}{\gamma var(er_{t+1}(1))} + \frac{1}{2}\frac{\gamma-1}{\gamma}\frac{cov(p_{t+1}(1) - p^*_{t+1}(1), er_{t+1}(1))}{var(er_{t+1}(1))} \\ &\quad + \frac{1}{2}\frac{(1-\psi')cov(f_{Ht+1}(1) - f_{Ft+1}(1), er_{t+1}(1))}{\gamma var(er_{t+1}(1))} \end{aligned} \tag{40}$$

where $er_{t+1} = r_{H,t+1} - r_{F,t+1}$ is the excess return, whose first-order component is $er_{t+1} = r_\epsilon \epsilon_{t+1}$ for a 1 by 3 vector r_ϵ that follows from the first-order solution. $f_{Ht+1}(1) = H_{1,H}S_{t+1}(1)$ is the first-order component of the function $f_H(S_{t+1})$. Finally, $\psi' = 1 - \beta(1-\psi)R(0)^{1-\gamma}$.

(40) shows that three channels can push investors away from a fully diversified portfolio, defined as $k(0) = 0.5$. The first reflects the cost of investing abroad, τ^2, with a higher cost making investing in domestic equity more attractive. The second channel reflects the co-movements of the real exchange rate and excess return. Assuming $\gamma > 1$, it is attractive for Home investors to invest in the Home equity if the excess return on Home equity is high in states where the Home price index is relatively high. The final channel captures a hedge against changes in future expected portfolio returns, which are captured by the functions $f_H(S_{t+1})$ and $f_F(S_{t+1})$ in the value function of Home and Foreign investors next period.

An increase in these functions imply a drop in welfare. It is attractive for Home investors to invest in Home equity when the excess return on Home equity is high in states where $f_H(S_{t+1})$ is relatively high.

Notice that each of the three components of $k(0)$ in (40) is a ratio of second-order variables. Both the numerator and the denominator of these terms are proportional to σ^2, and the ratio is therefore zero-order. This illustrates why the second-order components of portfolio Euler equations are necessary to compute the zero-order component of portfolio shares.

With the exception of τ^2, all the second-order components in the three ratios are based on variances and covariances of first-order components of model variables. These are based on the first-order solution (38)-(39). But the first-order solution (38)-(39) is in turn conditional on $k(0)$. This leads to a fixed point problem. We solve $k(0)$ as a fixed point of the function that maps $k(0)$ into itself: $k(0)$ maps into the first-order solution (38)-(39), which maps into $k(0)$ in (40). The solution described so far implements the solution algorithm in section 2 for $O = 1$.

4.3 The hard part

The final step is only necessary to compute gross external holdings or gross capital flows, which requires the first-order component of the portfolio share difference k_t^D. For a given average portfolio share, an increase in k_t^D implies that Home investors increase the share of Home equity in their portfolio and Foreign investors increase the share of Foreign equity.[15] Such a retrenchment reduces gross assets and liabilities. So far we have only solved for the zero-order component of k_t^D, which is $2k(0) - 1$.

In order to solve for the first-order component of k_t^D we need to implement the solution algorithm in section 2 for the case $O = 2$. It proceeds along the same line as the solution described above for $O = 1$, but now one order higher for all equations and variables. It is based on the third-order component of the difference in portfolio Euler equations, combined with the second-order components of all 10 "other" model equations. These are used to jointly solve for the first-order component of k_t^D and the second-order component of all "other" variables.

We start by solving the second-order component of the "other" variables conditional on a first-order solution for the portfolio share difference: $k_t^D(1) = k_s S_t(1)$,

[15] This follows because $k_{H,t}^H = k_t^A + 0.5k_t^D$, $k_{H,t}^F = k_t^A - 0.5k_t^D$ and $k_{F,t}^F = 1 - k_{H,t}^F$.

with k_s a 1 by 3 vector. The second-order components of the "other" variables are obtained after substituting the first-order solutions of all variables into the second-order components of the 10 "other" model equations. Since such second-order solutions are by now quite standard, we leave a full description of the algebra to the Technical Appendix.[16] For control variables, for example p_{Ft}, the solution takes the form

$$p_{F,t}(2) = p_s S_t(2) + S_t(1)' p_{ss} S_t(1) + k_p \sigma^2 \tag{41}$$

where p_s is a vector, p_{ss} a matrix and k_p a scalar. The second-order solution for state space accumulation takes the form

$$S_{t+1}(2) = N_1 S_t(2) + \begin{bmatrix} S_t(1)' N_{3,1} S_t(1) + \epsilon'_{t+1} N_{4,1} \epsilon_{t+1} + S_t(1)' N_{5,1} \epsilon_{t+1} \\ S_t(1)' N_{3,2} S_t(1) + \epsilon'_{t+1} N_{4,2} \epsilon_{t+1} + S_t(1)' N_{5,2} \epsilon_{t+1} \\ S_t(1)' N_{3,3} S_t(1) + \epsilon'_{t+1} N_{4,3} \epsilon_{t+1} + S_t(1)' N_{5,3} \epsilon_{t+1} \end{bmatrix} + N_6 \sigma^2 \tag{42}$$

where $N_{3,i}$, $N_{4,i}$ and $N_{5,i}$ are matrices and N_6 is a vector. Finally, Appendix B shows that the second-order component of the Bellman equations yield the second-order derivative of the functions $f_H(S_t)$ and $f_F(S_t)$ at the steady state.

In order to solve $k_t^D(1)$, we combine the second-order solution described above with the third-order component of the difference in portfolio Euler equations across countries. The latter is derived in Appendix C. The resulting first-order solution for k_t^D can be described as follows. For variables x and y with mean zero, define $\hat{var}(x) = Ex(1)x(2) + Ex(2)x(1)$ and $\hat{cov}(x, y) = Ex(1)y(2) + Ex(2)y(1)$. These capture respectively the third-order component of the variance of x and of the covariance between x and y. We normally think of variances and covariances as second-order terms, but to the extent that they are time varying, they include terms of of higher order.

The first-order solution for k_t^D from the third-order difference in portfolio Euler equations is

$$k_t^D(1) = -(2k(0) - 1) \frac{\hat{var}(er_{t+1})}{var(er_{t+1}(1))} + \frac{\gamma - 1}{\gamma} \frac{\hat{cov}(p_{t+1} - p^*_{t+1}, er_{t+1})}{var(er_{t+1}(1))} + \tag{43}$$
$$\frac{\hat{cov}\left(f_{Ht+1} - f_{Ft+1}, er_{t+1}\right) + 0.5\psi' E_t \left[(f_{Ht+1}(1))^2 - (f_{Ft+1}(1))^2 \right] er_{t+1}(1)}{\gamma var(er_{t+1}(1))/(1 - \psi')}$$

[16]For descriptions of second-order solutions see Kim et.al. (2003), Schmitt-Grohe and Uribe (2004) and Lombardo and Sutherland (2005).

In this expression the variance in the denominator of each ratio is second-order, while the terms in the numerator are all third-order, so that the ratios are all first-order. The three ratios capture respectively the time variation in the variance of the excess return, the time variation in the covariance between the real exchange rate and the excess return, and the time variation in the hedge against changes in expected portfolio returns. These same elements without their time variation are present in the zero order component (40) of portfolio shares.

An increase in the variance of the excess return by itself reduces home bias. For example, the higher expected return on domestic assets due to the transaction cost τ^2 on investment abroad translates into a smaller home bias the larger the variance of the excess return. This is captured by the first ratio in (43). An increase in the covariance between the real exchange rate and the excess return leads to increased home bias as it implies that for both Home and Foreign investors their domestic asset has a relatively high payoff when the domestic price index is high. This is captured by the second ratio in (43). Similarly, an increase in the covariance between the hedging term $f_H(S_{t+1}) - f_F(S_{t+1})$ and the excess return leads to increased home bias as it implies that for both Home and Foreign investors their domestic asset has a relatively high payoff when their utility is low.[17] This is captured by the last ratio in (43).

The third-order terms in the numerator of (43) take the form $\sigma^2 S_t(1)$. Since $var(er_{t+1}(1))$ in the denominator is proportional to σ^2, the ratio is proportional to S_t. This yields $k_t^D(1) = k_s S_t$ for a 1 by 3 vector k_s. We can then solve the vector k_s by solving the fixed point of a function that maps k_s into itself. For a given vector k_s we can solve the second-order components of the "other" model variables. Together with the first-order components of the "other" model variables it allows us to solve the time varying moments $v\hat{a}r$ and $c\hat{o}v$ in (43). This in turn yields a new vector k_s. Solving the fixed point problem yields the first-order solution of k_t^D.

[17] An increase in $f_H(S_{t+1})$ lowers utility for the Home country in time $t+1$ when $\gamma > 1$.

5 A numerical illustration

5.1 Parametrization

The implications of our simple model can be illustrated through a numerical example. The particular parameterization we adopt is for illustrative purposes only, not to match the data of any particular country. We believe that various extensions of the model will need to be introduced before it can be seriously confronted to the data.

We assume a labor share of output, θ, of 0.7. Productivity shocks are assumed to be highly persistent, with $\rho = 0.99$, and productivity innovations have a standard deviation of $\sigma = 5\%$. Turning to consumers' preferences, we assume home bias in preferences by setting $\alpha = 0.8$. The elasticity of substitution between Home and Foreign goods is set at $\lambda = 2$. The rate of relative risk-aversion,γ, is set at 10. Agents face a probability of death of $\psi = 0.05$, leading to a consumption-wealth ratio of 5%. The transaction cost on investing abroad, τ^2, is set at 0.4%. These parameters generate a sizable home bias in equity holdings, with $k(0) = 0.8$. We illustrate the dynamic response to a one standard deviation increase in Home productivity through nine charts.

5.2 Real exchange rate and equity prices

Chart 1 illustrates the dynamic response of the relative price of the Foreign good. The persistent increase in Home productivity boosts the supply of the Home good, leading to an increase in the relative price of the Foreign good by 2.6% on impact. This is followed by a gradual drop in the relative price for the Foreign goods towards its initial level as the shock gradually dissipates. The shock therefore leads to an immediate real depreciation of the Home currency, followed by a gradual appreciation.

Chart 2 shows the dynamic response of equity prices. It shows the Home equity price in units of the Home good (solid line) and the Foreign equity price in units of the Foreign good (dotted line). The persistent Home productivity shock immediately raises the Home equity price by 4.7%. The Foreign equity price rises by a small 0.3%. This is because the higher productivity boosts wealth, some of which is invested in Foreign equity. The increase in Foreign equity prices is larger when expressed in Home goods (2.9%), due to the increase in the relative price of Foreign goods. Yet Home equity prices still increase by more on impact. Following the initial jump, equity prices gradually drop back to their steady state. The expected drop in the Home equity price is clearly much larger than of the Foreign equity price. We return to this below when describing the role of expected valuation effects in the external adjustment process.

5.3 Financial positions

Chart 3 shows the dynamic response of gross external assets and liabilities of the Home country (dotted and solid lines), as well as its net external asset position (thick line). All are shown as a fraction of the initial GDP. Gross positions change both as a result of valuation effects and capital flows. It is therefore useful to view Chart 3 jointly with Chart 4, which shows net external assets (solid line) along with the cumulative net capital outflows (dotted line). The initial response of both gross assets and liabilities is almost entirely due to unexpected valuation effects. Chart 4 shows that initial net capital outflows are small in comparison. Gross liabilities rise due to the increase in the Home equity price that boosts the value of Foreign investors holding of Home equity. Gross assets rise both as a result of the rise in the Foreign equity price (in units of the Foreign good) and the large immediate real depreciation of the Home currency. Overall the net external position becomes negative at -6.1% of GDP. This reflects primarily the adverse valuation effect from high Home equity prices, which by itself leads to a -16.7% net external position. It is partially offset by the valuation gain due to the real depreciation of the Home currency, which by itself adds 10.0% to the net external position.

After the initial shock gross liabilities drop much faster than gross assets and soon the country becomes a net creditor. Chart 4 shows that this is driven to a large extent by cumulative net capital outflows, which are the result of an increase in savings by the Home country. On top of that the Home country also receives fully expected valuation gains that increase its net external position, reflecting to the gradual fall in Home equity prices in Chart 2. This is illustrated by the decreasing gap between cumulative capital outflows and the net external position in Chart 4.

5.4 Capital flows

Chart 5 shows the dynamic response of both gross and net capital flows as a fraction of the initial GDP. A positive gross capital outflow (solid line) captures purchases of Foreign equity by Home investors, while a positive gross capital inflow (dotted line) captures purchases of Home equity by Foreign investors. Net capital outflows (thick line) measure the difference between gross outflows and inflows. Initially there is a retrenchment in that both Home and Foreign investors reallocate portfolios towards domestic assets, leading to a drop in both gross capital outflows and inflows. Gross inflows drop by more, so that net capital outflows become positive. After that gross capital outflows and inflows almost perfectly mirror each other, with positive gross outflows and negative gross inflows. This means that both Home and Foreign investors are reallocating their portfolios towards Foreign equity by almost exactly the same amount after the initial shock, leading to continued net capital outflows.

The next three Charts illustrate the driving forces behind the gross and net capital flows in Chart 5. Chart 6 shows the portfolio share invested in Home equity by both Home (solid line) and Foreign (dotted line) investors. Chart 6 also shows the "passive portfolio share" (thick line) which reflects what the share would have been in the absence of capital flows. Without any asset trade, the increase in Home equity prices (Chart 2) automatically boosts the value of investors' holdings of Home equity, thereby raising the passive share of Home equity in portfolios.

Chart 6 shows that there is a difference between the Home and Foreign portfolio shares is positive. In the immediate response to the shock the Home portfolio share (0.30%) is a bit higher than the passive portfolio share (0.28%) and the Foreign portfolio share (0.24%) is a bit smaller. This means that Home investors actively reallocate their portfolio towards Home assets, as the increase in home equity prices still leave the passive portfolio share short of their desired share. Home investors then purchase additional home equity, accounting for the negative gross capital outflows in Chart 5. Foreign investors by contrast actively reallocate their portfolio towards Foreign assets, which accounts for the negative gross capital inflows in Chart 5. The difference in the direction of portfolio flows across the two countries is brought about by changes in the three elements in (43) affecting k^D: changes in the variance of the excess return, the covariance between the real exchange rate and the excess return and the hedge against changes in expected returns.

Chart 6 also shows that after the initial shock the portfolio shares of both Home and Foreign investors drop much faster than the passive portfolio share. This means that both Home and Foreign investors actively reallocate their portfolio towards Foreign assets, accounting for the positive gross capital outflows and the negative gross capital inflows in Chart 5. This reallocation towards Foreign equity by all investors is brought about by an increase in the expected excess return on Foreign equity. This change is third-order and therefore very small.[18] But expected excess returns are divided by the second-order variance of the excess return in the optimal portfolio, so that even a very small third-order component of expected excess returns can generate a first-order reallocation of portfolios. Note also that while changes in portfolio shares in Chart 6 may seem small, they can generate large capital flows when multiplied by total wealth.

The next two charts show that portfolio reallocation indeed accounts for most of the gross capital flows displayed in Chart 5. Some standard balance of payments accounting that we leave to the Technical Appendix shows that gross capital flows can be written as the sum of two components. The first is the active reallocation of the portfolio share from the passive portfolio discussed above. We call this the active *portfolio reallocation* effect. The second component of capital flows is a *portfolio growth* effect, which has been emphasized by Kraay and Ventura (2000). Holding constant the portfolio share at the steady state, an increase in national savings leads to capital outflows equal to the rise in national savings times the steady state fraction invested abroad.

Charts 7 and 8 document the breakdown of gross capital outflows and inflows (solid line) into the portfolio reallocation (thick line) and portfolio growth (dotted line) components. The shock leads to a rise in Home savings and an offsetting drop in Foreign savings (they add to zero). Therefore the portfolio growth effect by itself leads to positive capital outflows and negative capital inflows. While the portfolio growth effect is not negligible, for the assumed parameterization Charts 7 and 8 show that the portfolio reallocation effect dominates the overall dynamics of gross capital flows.

[18]Both the first and second order components of the expected excess return are zero in the model.

5.5 Channels of external adjustment

Our setup allows us to explore the channels through which the net external position of the Home country adjusts after the initial jump. Standard balance of payments accounting, reported in the Technical Appendix, implies that

$$GL_t(1) - GA_t(1) = \sum_{s=1}^{\infty} \frac{TA_{t+s}(1)}{R(0)^s} - GA(0) \sum_{s=1}^{\infty} \frac{r_{H,t+s}(1) - r_{F,t+s}(1)}{R(0)^{s-1}} \qquad (44)$$

where GA and GL represent gross assets and liabilities of the Home country and TA is the trade balance. (44) shows that the first-order component of net external debt is equal to the present value of future trade surpluses minus the present value of future excess returns. A net external debt can therefore either be financed by future trade surpluses or by more favorable future returns on external assets (Foreign equity) than external liabilities (Home equity).

Adding an expectation to both sides of (44), the net external debt is equal to the present value of *expected* future trade surpluses minus the present value of *expected* future excess returns. As expected future excess returns are zero to the first-order, this implies that the net external debt is simply equal to the present value of expected trade surpluses. The model can therefore not account for empirical findings by Gourinchas and Rey (2006) that net external debt is to some extent financed by differences in expected returns.

While the expected excess returns are zero, it is nonetheless of interest to look at their components. Breaking down asset returns into dividend yields and capital gains, we can write

$$GL_t(1) - GA_t(1) = \sum_{s=1}^{\infty} \frac{E_t\, TA_{t+s}(1)}{R(0)^s} + \sum_{s=1}^{\infty} \frac{E_t\left(nd_{t+s}(1) + nk_{t+s}(1)\right)}{R(0)^{s-1}} \qquad (45)$$

where nd_{t+s} captures the net dividend income associated with differences in dividend yields for the two assets and nk_{t+s} captures the difference in returns associated with capital gains and losses. The latter therefore captures expected valuation effects. It can further be broken down into valuation effects associated with expected changes in equity prices and expected changes in the real exchange rate.

Chart 9 breaks down the components of net external adjustment. In the immediate response to the shock the net external debt of the Home country reaches 6.1% of GDP. As discussed above, this is financed entirely through expected future trade surpluses, whose present value is also 6.1%. As Home productivity is persistently higher, the expected dividend yield is larger for Home than Foreign equity, which further increases the net external debt. In present value terms this adds 2.1% to the external debt. As the relative price of Foreign goods decreases (Chart 1), the Home real exchange rate is expected to appreciate, leading to a capital loss on Home investors' holdings of Foreign equity. In present value terms this channel adds 5.1% the the external debt. Finally, Home equity prices are expected to decrease faster that Foreign equity prices do (Chart 2), leading to an expected capital loss for Foreign investors on their holdings of Home equity. In present value terms, the expected valuation effects through changes in equity prices reduces the external debt of the Home country by 7.2%, exactly offsetting the impact of expected differences in dividend yields and expected real exchange rate appreciation.

6 Conclusion

We have developed a method for solving stochastic general equilibrium open-economy models of portfolio choice with the aim of better understanding the nature of international capital flows. The method has the advantage that it closely connects to existing first and second-order solution methods of stochastic general equilibrium models, while giving special treatment to optimality conditions for portfolio choice. It highlights the need to go to higher orders of these optimality conditions to solve for steady state and first-order components of the portfolio allocation and therefore capital flows.

The method also has the advantage that it can be broadly applied. The simple two country, two-asset, two-good example discussed in the paper illustrates what we can learn from such models. The next natural step is to extend this framework by introducing consumption and investment decisions. This will put us in a better position to confront the model to data on gross and net capital flows and make meaningful predictions related to the external adjustment process faced by countries with large external imbalances like the United States.

Ultimately we will also need to introduce monetary elements to address the call by Obstfeld (2004) for an "integrative general-equilibrium monetary model of international portfolio choice." Together with introducing fiscal policy and nominal bonds, this puts us into a better position to address policy questions.

Capital flows play a key role in discussions about the nature of exchange rate policy that countries should adopt. In the model welfare and capital flows are naturally intertwined as the second-order component of the value function requires the same solution method that gives the first-order component of gross and net capital flows.

Appendix

A Equations of the model

As discussed at the beginning of section 4, the model can be summarized by 11 equations. Writing variables other than portfolio shares in logarithmic form these equations are

$$a_{H,t+1} = \rho a_{H,t} + \epsilon_{H,t+1} \tag{46}$$

$$a_{F,t+1} = \rho a_{F,t} + \epsilon_{F,t+1} \tag{47}$$

$$\begin{aligned} e^{w_{t+1}+p_{t+1}} = {} & (1-\psi)\left[k_{H,t}^{H} e^{r_{H,t+1}} + (1-k_{H,t}^{H}) e^{r_{F,t+1}}\right] e^{w_t+p_t} + \\ & \theta e^{a_{H,t+1}} \end{aligned} \tag{48}$$

$$\begin{aligned} e^{w_{t+1}^{*}+p_{t+1}^{*}} = {} & (1-\psi)\left[k_{H,t}^{F} e^{r_{H,t+1}} + (1-k_{H,t}^{F}) e^{r_{F,t+1}}\right] e^{w_t^{*}+p_t^{*}} + \\ & \theta e^{p_{Ft+1}+a_{F,t+1}} \end{aligned} \tag{49}$$

$$e^{a_{H,t}} = \alpha\psi e^{w_t+\lambda p_t} + (1-\alpha)\,\psi e^{w_t^{*}+\lambda p_t^{*}} + \alpha\psi e^{w_t^{*}+\lambda\left(p_t^{*}-p_{F,t}\right)} \tag{50}$$

$$e^{q_{H,t}} = (1-\psi)\left[k_{H,t}^{H} e^{w_t+p_t} + k_{H,t}^{F} e^{w_t^{*}+p_t^{*}}\right] \tag{51}$$

$$e^{q_{F,t}} = (1-\psi)\left[k_{F,t}^{H} e^{w_t+p_t} + k_{F,t}^{F} e^{w_t^{*}+p_t^{*}}\right] \tag{52}$$

$$E_t\left((1-\psi)e^{v+f_H(S_{t+1})} + \psi\right) e^{-\gamma r_{t+1}^{p,H}} \left(e^{r_{H,t+1}} - e^{r_{F,t+1}-\tau^2}\right) e^{p_t-p_{t+1}} = 0 \tag{53}$$

$$E_t\left((1-\psi)e^{v+f_F(S_{t+1})} + \psi\right) E^{-\gamma r_{t+1}^{p,F}} \left(e^{r_{H,t+1}-\tau^2} - e^{r_{F,t+1}}\right) e^{p_t^{*}-p_{t+1}^{*}} = 0 \tag{54}$$

$$e^{v+f_H(S_t)} = \beta E_t\left((1-\psi)e^{v+f_H(S_{t+1})} + \psi\right) e^{(1-\gamma) r_{t+1}^{p,H}} \tag{55}$$

$$e^{v+f_F(S_t)} = \beta E_t\left((1-\psi)e^{v+f_F(S_{t+1})} + \psi\right) e^{(1-\gamma) r_{t+1}^{p,F}} \tag{56}$$

(46) and (47) are the autoregressive processes for productivity. (48)-(49) are the wealth dynamics in the Home and Foreign countries. (50) is the Home goods market clearing condition (we can omit the Foreign goods market clearing condition due to Walras's law). (51)-(52) are the market clearing conditions for Home and Foreign equities. (53)-(54) are the optimal portfolio conditions for Home and Foreign investors. Finally, (55)-(56) are the Bellman equations for Home and Foreign investors.

These equations depend on consumer price indices, asset and portfolio returns, which in logarithmic form can be written as

$$e^{(1-\lambda)p_t} = \alpha + (1-\alpha)\, e^{(1-\lambda)p_{Ft}} \tag{57}$$

$$e^{(1-\lambda)p_t^*} = (1-\alpha) + \alpha e^{(1-\lambda)p_{Ft}} \tag{58}$$

$$e^{r_{H,t+1}} = e^{q_{H,t+1}-q_{H,t}} + (1-\theta)e^{a_{H,t+1}-q_{H,t}} \tag{59}$$

$$e^{r_{F,t+1}} = e^{q_{F,t+1}-q_{F,t}} + (1-\theta)e^{p_{Ft+1}+a_{F,t+1}-q_{F,t}} \tag{60}$$

$$e^{r_{t+1}^{p,H}} = \left[k_{H,t}^{H} e^{r_{H,t+1}} + (1-k_{H,t}^{H})e^{r_{F,t+1}-\tau^2}\right] e^{p_t - p_{t+1}} \tag{61}$$

$$e^{r_{t+1}^{p,F}} = \left[k_{H,t}^{F} e^{r_{H,t+1}-\tau^2} + (1-k_{H,t}^{F})e^{r_{F,t+1}}\right] e^{p_t^* - p_{t+1}^*} \tag{62}$$

(57)-(58) define the consumer prices indexes. (59)-(60) define the rates of return on Home and Foreign equity. Finally, (61)-(62) define the rates of return on the portfolios of Home and Foreign investors.

B Expansions of the Bellman equation

The elements of the Bellmann equation for the Home investor (55) are solved by taking a second-order expansion around $S = 0$. The resulting expression contains both first- and second-order components. The first-order components are:

$$v\,(1) + H_{1,H} S_t\,(1) = (1-\psi')\left[v\,(1) + H_{1,H} E_t S_{t+1}\,(1)\right] + E_t(1-\gamma) r_{t+1}^{p,H}\,(1) \tag{63}$$

where $v\,(1)$ is the first-order component of v and $H_{1,H}$ is a 1x3 matrix with the first derivative of $f_H(S)$, evaluated at $S = 0$. ψ' is a transformation of the probability of death ψ: $\psi' = 1 - \beta(1-\psi)R(0)^{1-\gamma}$. (63) is solved by $v\,(1) = 0$ and:

$$H_{1,H} = (1-\gamma) r_s \left(I_3 - (1-\psi')N_1\right)^{-1}$$

where r_s is a 1x3 matrix taken from the first-order solution of the portfolio return for the Home investor from (38): $r_{t+1}^{p,H}\,(1) = r_s S_{t+1}\,(1)$, I_3 is a 3x3 identity matrix and N_1 is the 3x3 matrix from (39).

The second-order components of (55) are:

$$\begin{aligned} & H_{1,H} S_t\,(2) + \frac{1}{2}\left[\left[H_{1,H} S_t\,(1)\right]^2 + 2v\,(2) + S_t\,(1)' H_{2,H} S_t\,(1)\right] = \qquad (64) \\ & (1-\psi') H_{1,H} E_t S_{t+1}\,(2) + (1-\gamma) E_t r_{t+1}^{p,H}\,(2) + \frac{\psi'}{2} E_t\left[(1-\gamma) r_{t+1}^{p,H}\,(1)\right]^2 + \\ & \frac{1-\psi'}{2} E_t\left[\left[H_{1,H} S_{t+1}\,(1) + (1-\gamma) r_{t+1}^{p,H}\,(1)\right]^2 + 2v\,(2) + S_{t+1}\,(1)' H_{2,H} S_{t+1}\,(1)\right] \end{aligned}$$

where $v\,(2)$ is the second-order component of v and $H_{2,H}$ is a 3x3 matrix with the second derivative of $f_H(S)$, evaluated at $S = 0$.

(64) entails cross-products of the first-order components of the state variables, $S_{t+1}\,(1)$, and the portfolio return, $r^{p,H}_{t+1}\,(1)$. These terms are taken from the first-order solution (38)-(39). (64) also includes the second-order components of the state variables, $S_{t+1}\,(2)$, which are taken from (42), as well as the second-order component of the expected the portfolio return, $E_t r^{p,H}_{t+1}\,(2)$, which takes a from similar to (41):

$$E_t r^{p,H}_{t+1}\,(2) = r_s S_t\,(2) + S_t\,(1)'\, r_{ss} S_t\,(1) + \hat{r}\sigma^2 \tag{65}$$

where r_{ss} is a 3x3 matrix and $\hat{r}$ is a scalar.

We use (64), along with the solution for $S_{t+1}\,(1)$, $S_{t+1}\,(2)$, $r^{p,H}_{t+1}\,(1)$ and $E_t r^{p,H}_{t+1}\,(2)$ to solve for $H_{2,H}$.[19] The 9x1 vector $H^{vec}_{2,H}$ is the "vectorized" form of the 3x3 matrix $H_{2,H}$. Specifically, the first three elements of $H^{vec}_{2,H}$ are the first row of $H_{2,H}$, the next three elements are the second row of $H_{2,H}$ and the last three elements are the third row of $H_{2,H}$. $H^{vec}_{2,H}$ is solved from (64) as:

$$H^{vec}_{2,H} = (I_9 - (1-\psi')\hat{N})^{-1} H^{vec}_3$$

where I_9 is a 9x9 identity matrix. $\hat{N}$ is a 9x9 matrix that consists of cross-products of various elements of the N_1 matrix from (39). The 9x1 vector H^{vec}_3 is the "vectorized" form of a 3x3 matrix H_3. The matrix H_3 includes cross-products of the matrices $H_{1,H}$ and r_s, as well as the matrix r_{ss} in the second-order component of the expected portfolio return (65), specifically:

$$\begin{aligned} H_3 &= -H'_{1,H} H_{1,H} + 2F_1 + 2(1-\gamma) r_{ss} + (1-\psi') N'_1 H'_{1,H} H_{1,H} N_1 \\ &\quad + 2(1-\psi')(1-\gamma) N'_1 H'_{1,H} r_s + (1-\gamma)^2 r'_s r_s \\ F_1 &= (1-\psi') \sum_{v=1}^{3} H_{1,H}(v) N_{3,v} \end{aligned}$$

where $H_{1,H}(v)$ is the v'th element of the 1x3 vector $H_{1,H}$ and the 3x3 matrices $N_{3,v}$ are the same as in (42).

The corresponding matrices for the Foreign investor, $H_{1,F}$ and $H_{2,F}$, are computed analogously.

[19] We also solve for $v\,(2)$, but this element does not affect portfolio choice.

C First-order difference in portfolio shares

The solution of the first order component of the portfolio share difference k_t^D relies on the third-order components of the optimal portfolio conditions (53)-(54). The expansion of the condition for the Home investor (53) leads to:

$$\begin{aligned}
&E_t er_{t+1}(3) + E_t er_{t+1}(1) r_{t+1}^A(2) + E_t er_{t+1}(2) r_{t+1}^A(1) \\
&+E_t er_{t+1}(1)\left[(1-\psi')f_{Ht+1}(2) - \gamma r_{t+1}^{p,H}(2) + p_t(2) - p_{t+1}(2)\right] \\
&+E_t er_{t+1}(2)\left[(1-\psi')f_{Ht+1}(1) - \gamma r_{t+1}^{p,H}(1) + p_t(1) - p_{t+1}(1)\right] \\
&+\tau^2 E_t\left[r_{F,t+1}(1) + (1-\psi')f_{Ht+1}(1) - \gamma r_{t+1}^{p,H}(1) + p_t(1) - p_{t+1}(1)\right] + O_3 = 0
\end{aligned} \tag{66}$$

where $er_{t+1}(i) = r_{H,t+1}(i) - r_{F,t+1}(i)$, $r_{t+1}^A(i) = 0.5\left[r_{H,t+1}(i) + r_{F,t+1}(i)\right]$, and τ^2 is second-order. The first term in (66) is the third-order component of the expected excess returns. The next two terms are the third-order components of the cross-product between excess returns and the average return, and consists of products of first- and second-order terms. Similarly, the fourth and fifth terms are the third-order components of the cross-product between excess returns and the pricing kernel. The sixth term reflects the friction in investing abroad, τ^2. The last term in (66) consists of cubic-products of first-order elements:

$$\begin{aligned}
O_3 \;=\; & \frac{1}{6}E_t\left[(r_{H,t+1}(1))^3 - (r_{F,t+1}(1))^3\right] \\
&+E_t\left[(1-\psi')f_{Ht+1}(1) - \gamma r_{t+1}^{p,H}(1) + p_t(1) - p_{t+1}(1)\right] r_{t+1}^A(1)\, er_{t+1}(1) \\
&+\frac{1}{2}(1-\psi')E_t\left[f_{Ht+1}(1) - \gamma r_{t+1}^{p,H}(1) + p_t(1) - p_{t+1}(1)\right]^2 er_{t+1}(1) \\
&+\frac{1}{2}\psi' E_t\left[-\gamma r_{t+1}^{p,H}(1) + p_t(1) - p_{t+1}(1)\right]^2 er_{t+1}(1)
\end{aligned}$$

The various components of O_3 are solved using the first-order solution (38)-(39). We can show that the resulting expression is:

$$O_3 = 2r_{DE}B_H A_H \sigma^2 S_t + \frac{\psi'(1-\psi')}{2}E_t\left[f_{Ht+1}(1)\right]^2 er_{t+1}(1)$$

where A_H is a 1x3 vector and r_{DE} and B_H are scalars. r_{DE} reflects the sensitivity of the first-order excess return to innovations:

$$er_{t+1}(1) = r_{DE}\epsilon_{t+1}^D$$

where $\epsilon^D_{t+1} = \epsilon^H_{t+1} - \epsilon^F_{t+1}$. A_H and B_H reflect the first-order solution of a combination of the average rate of return $r^A_{t+1}(1)$ and the pricing kernel:

$$r^A_{t+1} + (1-\psi')f_{Ht+1}(1) - \gamma r^{p,H}_{t+1}(1) + p_t(1) - p_{t+1}(1) = A_H S_t + B_H \epsilon^D_{t+1} + C_H \epsilon^A_{t+1}$$

where C_H is a scalar and $\epsilon^A_{t+1} = 0.5\left(\epsilon^H_{t+1} + \epsilon^F_{t+1}\right)$.

We undertake similar steps using the condition for the Foreign investor (54). Combining the resulting expression with our results, we write:

$$\begin{aligned}
&\frac{\psi'(1-\psi')}{2} E_t\left[[f_{Ht+1}(1)]^2 - [f_{Ft+1}(1)]^2)\right] er_{t+1}(1) \qquad (67)\\
&+E_t er_{t+1}(1)\left[\begin{array}{c} (1-\psi')\left[f_{Ht+1}(2) - f_{Ft+1}(2)\right] - \gamma\left(r^{p,H}_{t+1}(2) - r^{p,F}_{t+1}(2)\right) \\ +\left(p_t(2) - p^*_t(2)\right) - \left(p_{t+1}(2) - p^*_{t+1}(2)\right) \end{array}\right]\\
&+E_t er_{t+1}(2)\left[\begin{array}{c} (1-\psi')\left[f_{Ht+1}(1) - f_{Ft+1}(1)\right] - \gamma\left(r^{p,H}_{t+1}(1) - r^{p,F}_{t+1}(1)\right) \\ +\left(p_t(1) - p^*_t(1)\right) - \left(p_{t+1}(1) - p^*_{t+1}(1)\right) \end{array}\right] = 0
\end{aligned}$$

The first-order component of the difference in portfolio shares, $k^D_t(1)$, enters (67) through the second-order components of the portfolio returns. Taking the second-order components of (61)-(62) leads to:

$$\begin{aligned}
&r^{p,H}_{t+1}(2) - r^{p,F}_{t+1}(2) = (2k(0) - 1)\, er_{t+1}(2) + \left(p_t(2) - p^*_t(2)\right)\\
&- \left(p_{t+1}(2) - p^*_{t+1}(2)\right) + k^D_t(1)\, er_{t+1}(1)
\end{aligned}$$

Similarly, taking the first-order components of a first-order expansion of (61)-(62) leads to:

$$r^{p,H}_{t+1}(1) - r^{p,F}_{t+1}(1) = (2k(0) - 1)\, er_{t+1}(1) + \left(p_t(1) - p^*_t(1)\right) - \left(p_{t+1}(1) - p^*_{t+1}(1)\right)$$

Using this result, (67) becomes:

$$\begin{aligned}
&\frac{\psi'(1-\psi')}{2} E_t\left[[f_{Ht+1}(1)]^2 - [f_{Ft+1}(1)]^2)\right] er_{t+1}(1)\\
&+(1-\psi')\hat{cov}\left(f_{Ht+1} - f_{Ft+1}, er_{t+1}\right) - \gamma(2k(0) - 1)\hat{var}(er_{t+1})\\
&+(\gamma - 1)\,\hat{cov}(p_{t+1} - p^*_{t+1}, er_{t+1}) - \gamma k^D_t(1)\, var(er_{t+1}) = 0 \qquad (68)
\end{aligned}$$

where $\hat{cov}\left(x_{t+1}, y_{t+1}\right) = E_t x_{t+1}(1)\, y_{t+1}(2) + E_t x_{t+1}(2)\, y_{t+1}(1)$ and $\hat{var}\left(x_{t+1}\right) = \hat{cov}\left(x_{t+1}, x_{t+1}\right)$ and $var(er_{t+1}) = E_t\left[er_{t+1}(1)\right]^2$. (43) follows simply from (68).

The elements of (68) are computed by using the first-order solution (38)-(39), the second-order dynamics of the state variables, (42), and the second-order solution for the control variables, which are of the form of (41). For instance, the excess returns are:

$$er_{t+1}(1) = r'_{\epsilon}\epsilon_{t+1} \qquad\qquad er_{t+1}(2) = S_t(1)' M\epsilon_{t+1}$$

where r'_{ϵ} is a 1x2 vector, $\epsilon_{t+1} = \left[\epsilon^H_{t+1}, \epsilon^F_{t+1}\right]'$ and M is a 3x2 matrix. Using these expression, we write:

$$v\hat{a}r(er_{t+1}) = 2E_t er_{t+1}(1)\, er_{t+1}(2) = 2\sigma^2 r'_{\epsilon} M' S_t(1) \tag{69}$$

(69) shows that the third-order components of the variances and covariances in (68) reflect the second-order variance of the innovations, σ^2, along with the first-order state variables, $S_t(1)$. Solving for all the third-order components of the variances and covariances in (68) along similar lines we compute the first-order difference in portfolio shares as a function of the first-order state variables:

$$k^D_t(1) = k_s S_t(1)$$

where k_s is a 1x3 vector.

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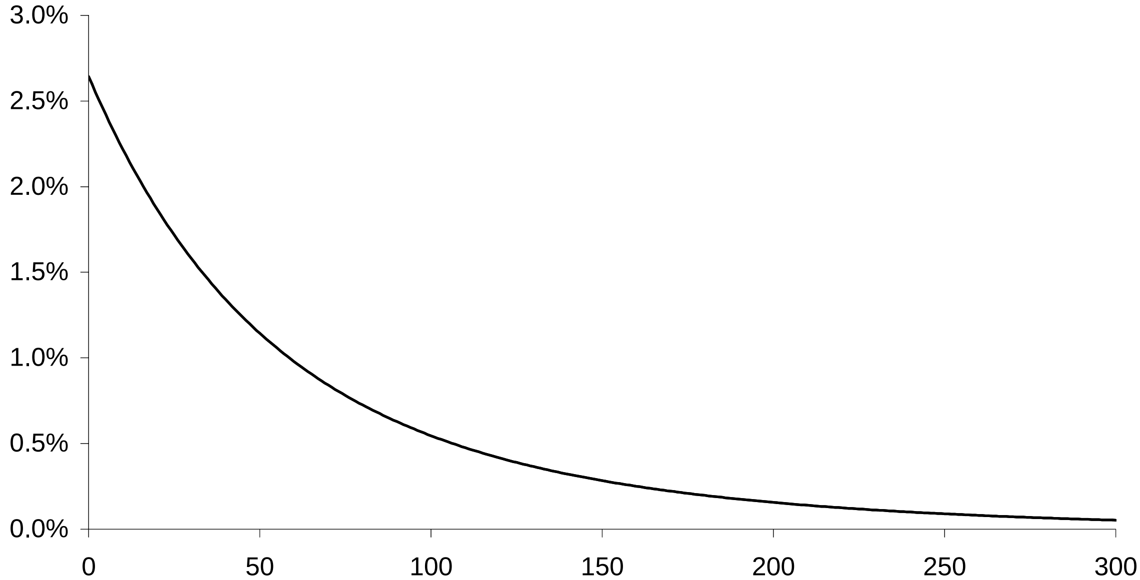
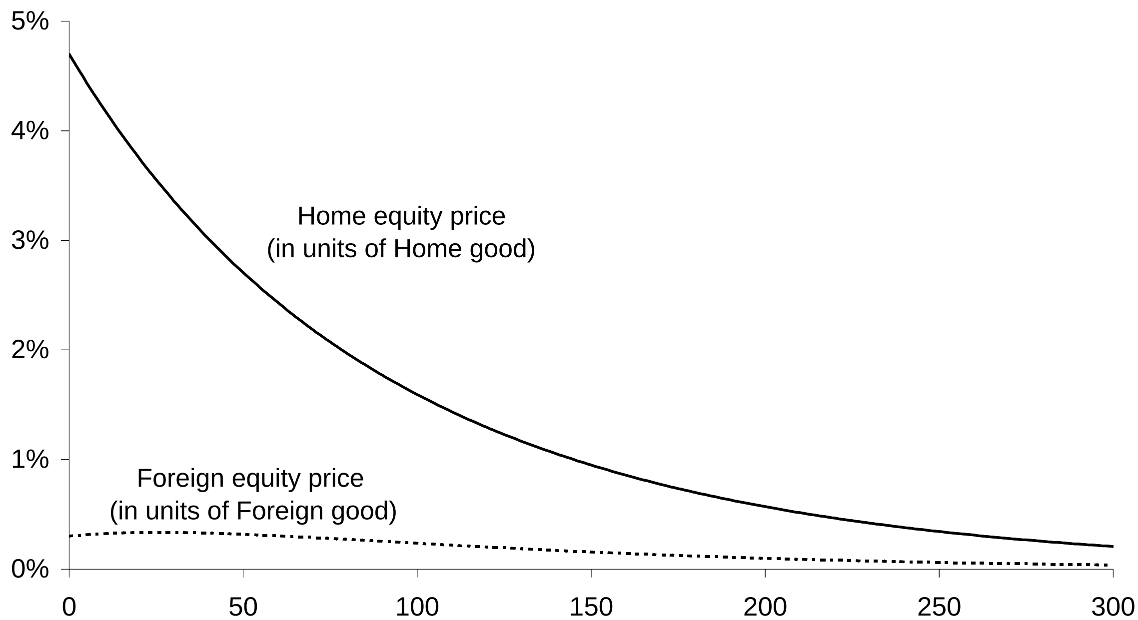
Chart 1: Relative price of the Foreign good*



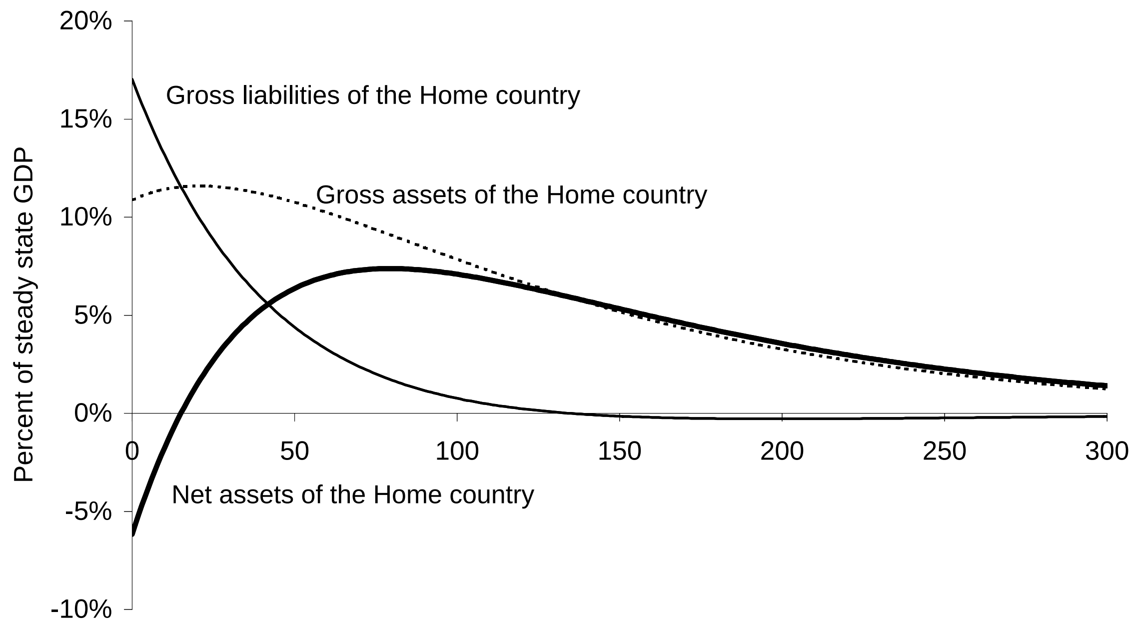
* Impulse response of the relative price of the Foreign good to a 5% increase in Home productivity. An increase in the relative price of the Foreign good corresponds to a real depreciation for the Home country.

Chart 2: Equity prices*



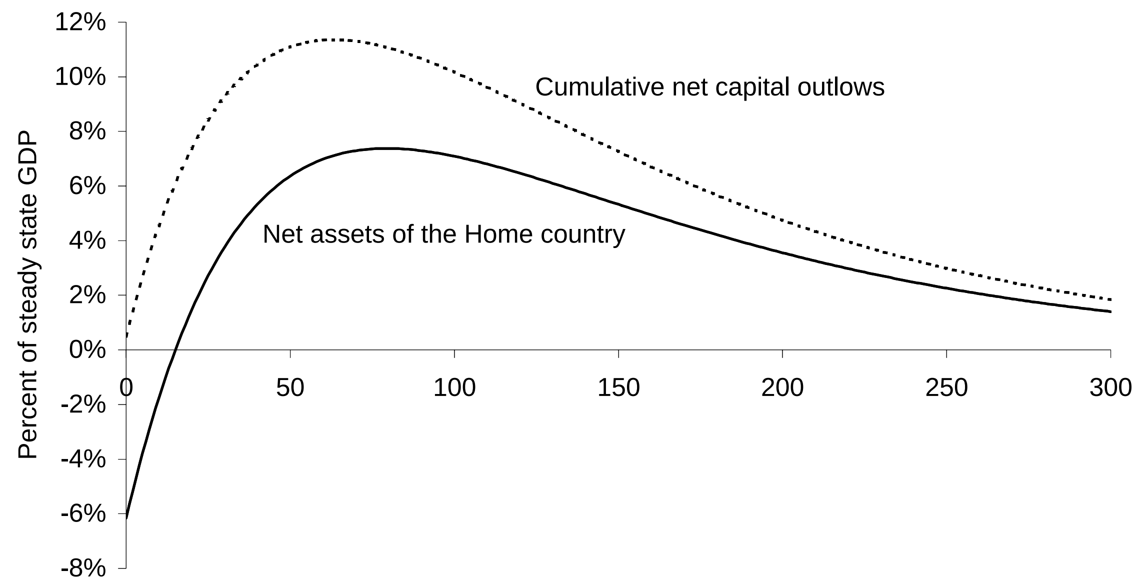
* Impulse response after a 5% increase in Home productivity. In terms of the notation in the text the lines represent q_H and q_F-p_F.

Chart 3: International assets and liabilities*



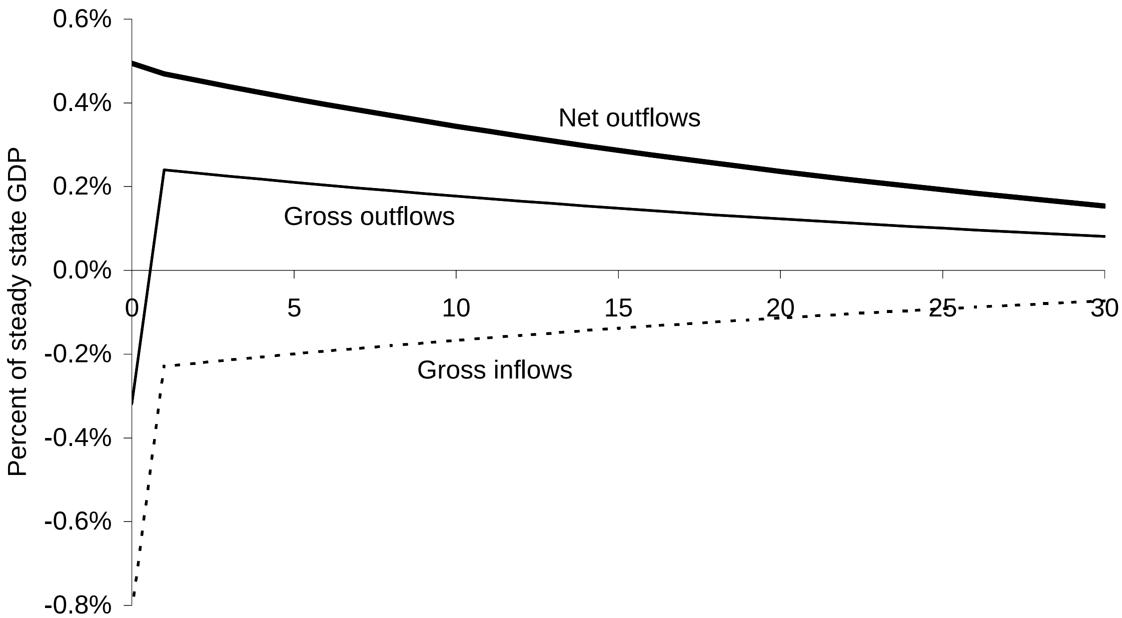
*Impulse response after a 5% increase in Home productivity.The gross assets of the Home country are the gross liabilities of the Foreign country.

Chart 4: Net assets and cumulative net capital outflows*



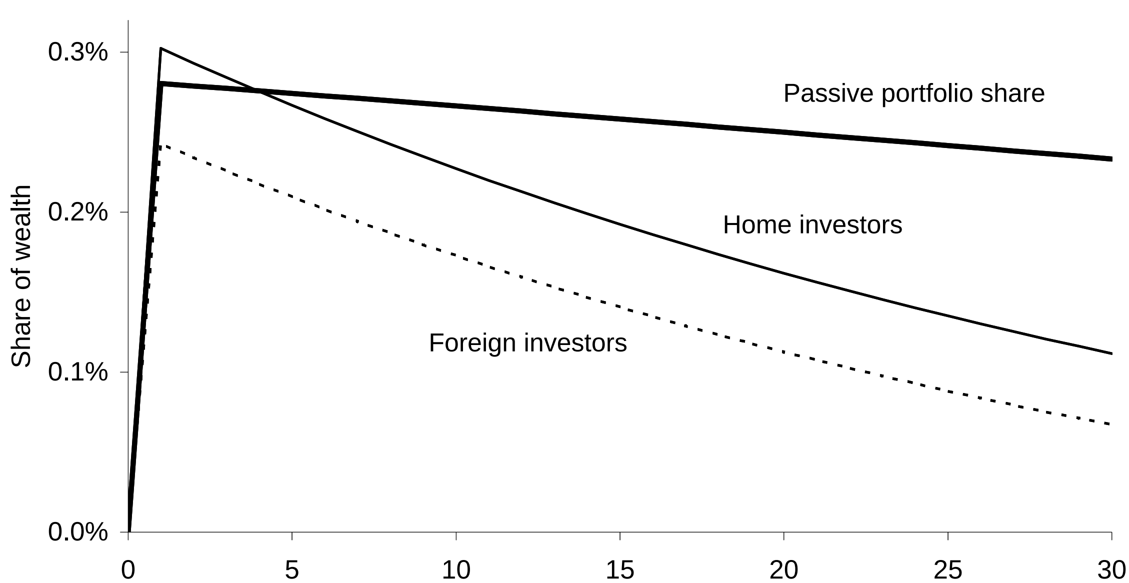
* Impulse response after a 5% increase in Home productivity. The 'cumulative net capital outflows' line at period t denotes the sum of net capital outflows from Home to Foreign between period zero and period t.

Chart 5: Gross and net capital flows*



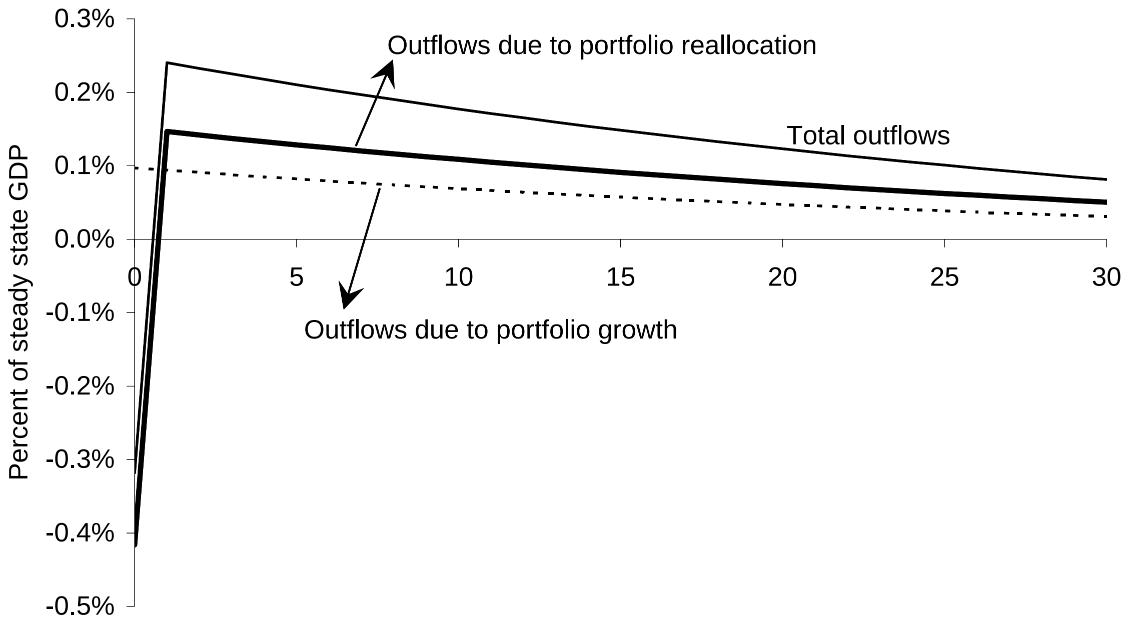
* Impulse response after a 5% increase in Home productivity. Positive values for gross outflows indicate a purchase of Foreign equity by Home investors. Positive values for gross intflows indicate a purchase of Home equity by Foreign investors.

Chart 6: Share of Home equity in portfolio*



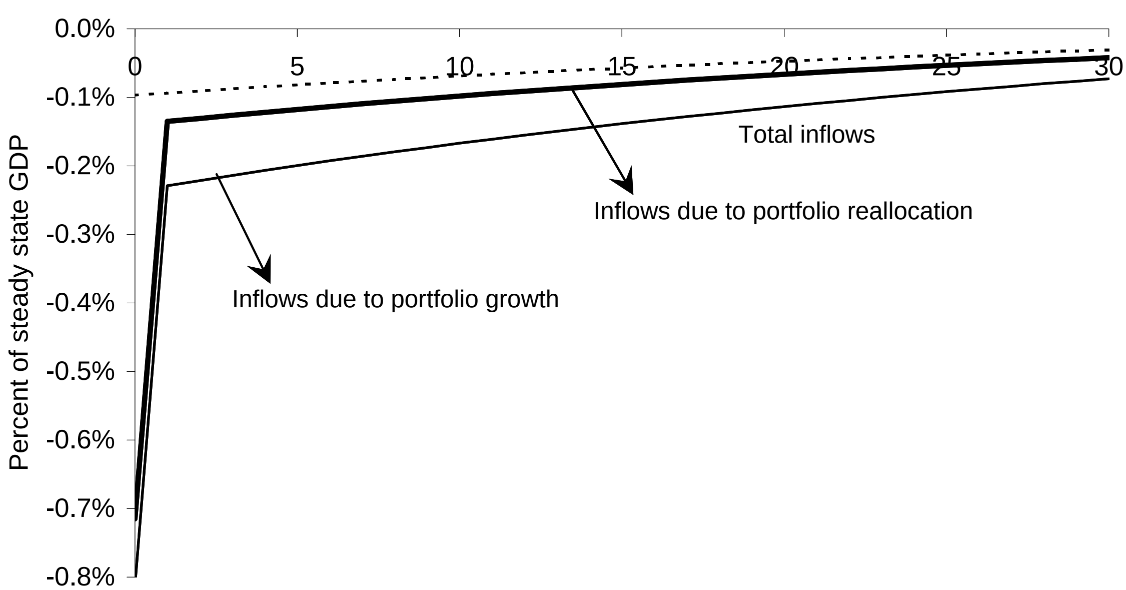
* Impulse response after a 5% increase in Home productivity. The chart shows the change in the share invested in Home equity. The passive portfolio share reflects the direct impact of movements in equity prices (the change in the portfolio share without equity trade).

Chart 7: Breakdown of gross capital outflows*



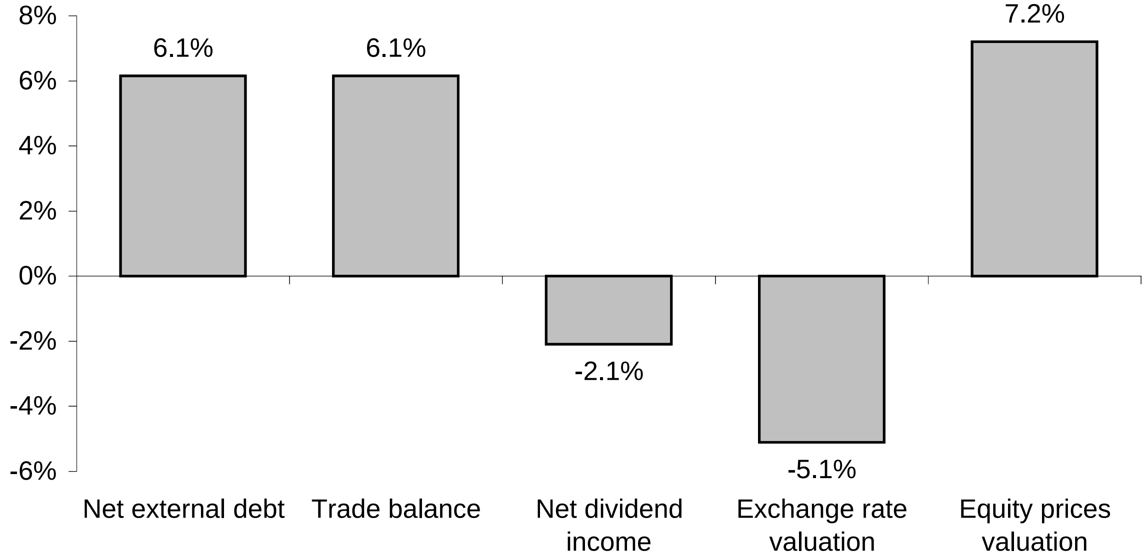
* Portfolio reallocation indicates capital outflows due to active reallocation towards Foreign equity. Portfolio growth indicates capital outflows due to increased saving, allocated across assets at steady state portfolio shares.

Chart 8: Breakdown of gross capital inflows*



* Portfolio reallocation indicates inflows due to reallocation towards Home equity. Portfolio growth indicates inflows due to increased saving, allocated at steady state portfolio shares. Both are negative as Foreign saving drops and the portfolio is reallocated to Foreign equity.

Chart 9: External adjustment channel: net present values*



* Changes after 5% increase in Home productivity. All values are measured at the end of the period when the shock occurs, after any initial jump in response to the shock. The 'net external debt' column indicates the value of the Home net external debt as a fraction of GDP. The 'trade balance' column is the net present value of expected future trade surpluses of the Home country. The 'net dividend income' column is the present value of expected net dividend income of the Home country (negative value=expected positive net dividend payments to Foreign country). The 'exchange rate valuation' column is the present value of expected future valuation gains due to a real depreciation of Home currency (negative value=valuation losses due to expected real appreciation). The 'equity prices valuation' column is the net present value of expected future valuation gains due to equity price changes.