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Chi-Fai Lo and Cho-Hoi Hui

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Pricing Corporate Bonds With Interest Rates Following Double Square-root Process

Chi-Fai Lo[+]
The Chinese University of Hong Kong

and

Cho-Hoi Hui[*,++]
Hong Kong Monetary Authority

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Abstract

This paper develops a corporate bond pricing model following the structural approach in which the dynamics of the instantaneous risk-free interest rate are governed by a double square-root (DSR) process. Credit spreads generated from this pricing model depend explicitly upon the levels of interest rates via a nonlinear effect arising from a DSR process. Given a positive correlation between interest rates and leverage ratios, the credit spreads generated by this pricing model have a negative relationship with interest rates, that is consistent with empirical findings using bond market data over the period 2008-2013 when interest rates were low.

JEL Classification: G13; G21; G28

Keywords: Corporate bond pricing model; Stochastic interest rate; Leverage ratio

[+] Institute of Theoretical Physics and Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong SAR, China. Email: cflo@phy.cuhk.edu.hk Tel: (852)39436362

[*,++] Corresponding author. Research Department, Hong Kong Monetary Authority and Hong Kong Institute for Monetary Research, 55/F, Two International Finance Centre, 8 Finance Street, Central, Hong Kong SAR, China. Email: chhui@hkma.gov.hk Tel: (852)28781485

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1. Introduction

Black and Scholes (1973) and Merton (1974) have pioneered using a contingent-claims framework for pricing corporate bonds. In Black-Scholes-Merton's structural framework, the capital structure is explicitly considered and default happens if the total asset value is lower than the value of liabilities at a bond’s maturity. Default risk is therefore equivalent to a European put option on a firm's asset value. The term structure of risk-free interest rates in the framework is the drift term in the asset value process and is assumed to be constant. Because of these properties of the structural model, there are two factors driving corporate bond values. The first factor is the probability that the issuer will default on their obligation. To have more accurate measures of default probability, subsequent studies mainly focus on the liability structure, and models with complex and dynamic liability structures including Jones et al. (1984), Leland (1998), Tauren (1999), Colin-Dufresne and Goldstein (2001), and Hui et al. (2003) have been developed.

The second factor is the required rate of return on default-free debt. An important concern is how to incorporate stochastic risk-free interest rates into a structural pricing model. To address this concern, general equilibrium term-structure models of risk-free interest rates such as the Vasicek (1977) model and Cox–Ingersoll–Ross (CIR) model (1985) are conventionally used in the two-factor corporate bond pricing framework. The term-structure models assume that short-term interest rates are derived from general assumptions about the state variables which describe the overall economy. Using the assumed process for short-term rates, one can determine the yield on longer-term bonds and the associated bond price solutions (usually in closed-form) based on the expected path of interest rates until the maturity of the bonds under a no-arbitrage condition.

The corporate bond pricing models proposed by Shimko et al. (1993), Longstaff and Schwartz (1995), and Colin-Dufresne and Goldstein (2001) specify that a firm’s asset value is follows the drift of risk-free interest rates governed by the Vasicek model following the Ornstein-Uhlenbeck process. With the Black-Scholes-Merton framework in Shimko et al. (1993) and constant liability in the Longstaff and Schwartz model (i.e., a fixed default barrier), the models predicts that the expected liability-to-asset ratio (i.e., leverage ratio) will decline exponentially over time. In addition to the asset side, the Colin-Dufresne and Goldstein model specifies the drift of a mean-reverting default barrier (i.e., a firm's liability) to be a decreasing function of interest rates, which amplifies the static effect of interest rates on the leverage ratio. Kim et al. (1993) propose a pricing model that incorporates the CIR model for interest rates following a mean-reverting square-root process.

One unresolved issue arising from the use of the CIR and Vasicek models is the relation between risk-free interest rates and credit spreads. Due to the specifications in the structural models proposed by Shimko et al. (1993), Kim et al. (1993), Longstaff and Schwartz (1995), and Colin-Dufresne and Goldstein (2001) in which a firm's asset value is drifted by risk-free interest rates, these models predict a negative relation between risk-free interest rates and credit spreads. However, the specification of the asset value being drifted by interest rates also predicts that the expected leverage ratio will decline exponentially over time. The decline in the expected leverage ratio is however not supported by empirical observations.[1] In addition, empirical studies have found mixed evidence on the relation between risk-free interest rates and credit spreads.

Using Moody's corporate bond yield indices, Longstaff and Schwartz (1995) find that a negative correlation between interest rates and credit spreads persists after controlling for stock returns. Collin-Dufresne et al. (2001) also find a negative relation between them after controlling for both firm- and market-level determinants of default risk. Similarly, Campbell and Taskler (2003) and Avramov et al. (2007) show a negative relation between the level or changes in spreads with interest rates after controlling for idiosyncratic firm-level risk, and aggregate and firm-level variables. Using Australian Eurobonds, Batten et al. (2005) find credit spreads negatively related to changes in Australian Government bond yields (i.e., a proxy for risk-free interest rates). Duffee (1998) demonstrates that credit spreads of both investment-grade callable and noncallable bonds fall when the 3-month Treasury bill yield increases, but the extent decreases with the initial credit quality of the bonds. The negative relationship between credit spreads and risk-free interest rates is much stronger for callable bonds. Ericsson and Renault (2006) also find a negative relation between interest rates and credit

[1] To correct this defect, Briys and de Varenne (1997) develop a corporate bond pricing model which is characterized by both the asset value and default barrier following the risk-free interest rate under a Gaussian diffusion process. As a result, the default barrier is assumed to grow with the firm value over time and the expected level of leverage is constant. The effect of the interest rate is thus only mediated via the parameters governing the stochastic dynamics of the interest rate.

spreads after controlling for liquidity risk in the Treasury bond market.

On the other hand, Schaefer and Strebulaev (2008) show that the relation between credit spreads and interest rates is much weaker than the theoretical predictions, using corporate bond data covering the period from December 1996 to December 2003. By using cointegration to model corporate credit and Treasury yields, Morris et al. (2000) find that an increase in Treasury yields causes credit spreads to narrow in the short-run, but over the long-run higher yields cause spreads to widen. Similarly, Miloudi and Moraux (2009) demonstrate that Treasury yields impact the short run dynamics of credit spreads but the long run relation has only a limited effect. Jacoby et al. (2009) use Canadian bond data (after removing tax effects) to show that the relation between interest rates and corporate spreads is negligible. Wu and Zhang (2008) and Davis (2008) both find that the relation between interest rates and credit spreads is partially determined by inflation.

Given that the specification of the asset value or liability being drifted by interest rates under the CIR and Vasicek models is not supported by empirical evidence, this paper proposes a two-factor corporate bond pricing model by incorporating a stochastic interest rate under a double square-root (DSR) process as proposed by Longstaff (1989). One important characteristic of the DSR model is that it has a nonlinear restoring force in its drift term. The leverage ratio in the proposed corporate bond pricing model follows a mean-reverting lognormal process without any interest-rate drift.[2] We derive closed-form solutions of corporate bond prices with default at maturity and default prior to maturity with an adjustable moving default barrier respectively. The credit spreads obtained from the model depend explicitly upon the level of the interest rate via a nonlinear effect arising from nonlinearity in the drift term of the DSR process, if the correlation between the interest rate and leverage ratio is non-zero. Given a positive correlation between the interest rate and leverage ratio, credit spreads decrease with an increase in the interest rate. However, if the interest rate and leverage ratio is negatively correlated, credit spreads increase with the interest rate. The proposed model is therefore able to capture different relationships between credit spreads and interest rates as demonstrated by the mixed empirical findings.

[2] The specification of a leverage ratio governed by a mean-reverting lognormal process is more general than for a lognormal process and consistent with the empirical studies, including Marsh (1982), Jalilvand and Harris (1984), Auerbach (1995) and Opler and Titman (1995), which document that companies tend to gradually adjust their capital structure towards a target level of leverage. The findings call for the mean-reverting-leverage model for pricing corporate bonds, as in Collin-Dufresne and Goldstein (2001).

If the CIR model or Vasicek model instead of the DSR model is used for the interest rate dynamics in the proposed corporate bond pricing framework, the effect of the interest rate on credit spreads generated by the framework is mediated via the parameters governing the stochastic dynamics of the interest rate [see the corporate bond pricing solutions using the Vasicek model of Eq. (11) in Briys and de Varenne (1997) and Eq. (9) in Hui et al. (2003)] but not the level of the interest rate explicitly. The difference is due to the characteristic of the drift terms with a linear restoring force in the CIR and Vasicek models, which is different from the nonlinear restoring force in the drift term of the DSR model. Using the DSR model, the model-generated credit spreads depend explicitly upon the level of the interest rate arising from the nonlinearity in the drift term of the interest rate model. This feature is consistent with some empirical findings including those presented in this paper.

This paper shows a nonlinear term arising from the DSR process of the interest rate in the corporate bond pricing solution of the proposed structural model. The nonlinear term links up the dynamics of the leverage ratio and the interest rate such that the bond credit spread depends explicitly upon a stochastic risk-free interest rate. Such dependence can be identified as a feature of the structural model using the leverage ratio as a factor driving corporate bond prices. Similar to the framework developed by Duffie and Lando (2001), if the dynamics of the leverage ratio with imperfect accounting information is treated as a hazard rate process which governs an inaccessible default stopping time, the proposed structural model can be recast as a special case of the second approach to pricing corporate bonds called “reduced-form models”. Furthermore, with some modifications of the default boundary condition such as a random default barrier, the default stopping time also becomes inaccessible.[3] However, other reduced-form models may not have the same dependence due to their model assumptions. For example, Jarrow and Turnbull (1995) assume that the hazard rate process and interest rate process are mutually independent. Therefore, the credit spread does not depend on the level of the interest rate given any specification of the interest rate process. Duffie and Singleton (1999) propose a reduced-form model in which the usual default-risk-free interest rate is replaced by a default-adjusted short rate process composed of the hazard rate. To have credit spreads depending explicitly upon the level of the interest rate, further assumptions need to be made for the model

[3] Another assumption is that the payoff upon default is specified exogenously. Jarrow and Protter (2004) illustrate and compare the mathematical structure of structural and reduced-form models.

specifications between the interest rate and hazard rate. As there is no link suggested by the literature on the economics of firm behaviour, the event of default and the interest rate in the model, calibration of the hazard rate process will be further complicated, and makes the implementation more difficult than the proposed model in this paper.

Using S&P’s US dollar domestic composite curves and Moody’s Long-Term Corporate Bond Yield Averages covering the low interest rate period from January 2008 to June 2013, we find that credit spreads are negatively related to the level of the interest rate, and the effect of the interest rate on credit spreads is economically significant. This finding is consistent with the implications of our proposed model where the correlation between the interest rate and leverage ratio is positive, which is supported by the observation that firms underwent a deleveraging process during the low interest rate environment caused by an extremely weak economy after the global financial crisis in 2008. This demonstrates the importance of the risk-free interest rate dynamics in addition to default risk in valuing corporate debt securities. In particular, the results provide supporting evidence for the empirical implications of incorporating the DSR model of the interest rate into the proposed corporate bond pricing model. The primary advantage of this model is that its closed-form solution is easily used to provide specific pricing and hedging results for corporate bonds rather than just general implications. The model also provides a simple theoretical benchmark against which the observed properties of corporate bond prices can be compared in a low interest rate environment.

Another feature of the DSR interest rate model is that it is able to capture the behaviour of the short-term interest rate in a low interest rate environment. The nonlinear drift term in the DSR model allows the interest rate to behave in a sticky way near the zero boundary. The model is thus applicable to a low interest rate environment in which the short-term interest rate tends to persist near the zero bound, as shown during the 1930-1940 period (see table A1 in Cecchetti (1988)) and the global financial crisis since 2008, instead of moving back towards higher levels in a short time as implied by the CIR and Vasicek models. The global financial crisis followed the sub-prime crisis in the US in mid-2007. Following the bankruptcy of Lehman Brothers in mid-September 2008, developments took a dramatic turn and spilled over to other economies. During 2008, the US Fed lowered the policy interest rate from 4% level to 0-0.25% to provide monetary support for the economy. Subsequently, it has taken unprecedented measures including quantitative easing policies to lower long-term borrowing costs and foster economic activity. As the interest-rate term structure is affected by the Federal Reserve's ultra-accommodative monetary policy, the 3-month US Treasury-bill yield has fallen to near zero for extended periods of time, as shown in Figure A1, and the 10-year Treasury yield has declined, hitting an historical low of 1.39% in July 2012. Therefore, we argue that the DSR model is more appropriate than the CIR and Vasicek models for pricing corporate bonds in the current low interest rate environment.

There are some empirical findings supporting the DSR model. The empirical results in Longstaff (1989) suggest that by estimating the model parameters, the DSR model is more successful in capturing the level and variation of 6- to 12-month Treasury bill yields during the 1964-1986 period compared with the CIR model. The results also suggest that yields are nonlinearly related to the risk-free interest rate as the model implies. Ahn et al. (2002) test the empirical performance of the quadratic term structure models including the DSR model in explaining historical bond price behaviour in the US during the period December 1946 – February 1991. They find that the quadratic term structure models outperform the affine term structure models including the CIR and Vasicek models by adding more flexibility to better match conditional moments of yields and matching correlations across yields. Similar resuts are found by Leippold and Wu (2003), and Li and Zhao (2006). Aït-Sahalia (1996) shows that there is evidence of nonlinearity in the drift function of the interest rate term structure by using a nonparametric approach. During the period of Japan's near-zero short-term rates in 2001-2005, Kim and Singleton (2012) find that the quadratic dynamic term structure models capture some of the key features of the Japan Government Bond data, including the variation in bond yields, and bond risk premiums that are very small when the level of interest rates is low. Recently, Ang et. al. (2012) estimate the effect of shifts in monetary policy in the US using the quadratic term structure model of interest rates. They find that long-term bonds are priced by agents who care about shifting monetary policy risk.

The scheme of this paper is as follows. In the following section we develop a corporate bond pricing model under a structural-model framework with interest rates governed by a DSR process. The characteristics of credit spreads calculated from the model are discussed in section 3. In section 4, we study the empirical relationship between credit spreads and the risk-free interest rate since 2008. The final section concludes.

2. Corporate bond pricing model

We assume a continuous-time framework for the valuation of corporate bonds in which the short-term interest rate and firm value are stochastic variables. The dynamics of the risk-free short-term interest rate *r* at time *t* is drawn from a term structure model governed by a DSR process, along the lines of Longstaff (1989):[4]

$$dr = \kappa_r \left(\theta_r - \sqrt{r}\right) dt + \sigma_r \sqrt{r} dz_r \qquad (1)$$

where $\kappa_r, \sigma_r > 0$ and , $\theta_r = \sigma_r^2 / 4\kappa_r > 0$. Eq. (1) is written using an historical probability measure. The drift term in Eq. (1) which is based on historical measures is a nonlinear restoring force which makes the dynamics of the interest rate different from those in the CIR and Vasicek models which have a linear restoring force. While the DSR model and the CIR model have a number of common empirically relevant characteristics, such as negative interest rates being precluded, and have a stationary distribution, the DSR model has two particular features due to the nonlinear restoring force.[5] First, only two parameters κ_r and σ_r^2 are required to determine the interest-rate dynamics. This is because θ_r^2 which is the long-run interest rate, is a function of the other two parameters such that $\theta_r^2 = \sigma_r^4 / 16\kappa_r^2$. Second, the interest rate is sticky downward as illustrated by Longstaff (1989) (see Appendix A for a discussion of the properties of the DSR model).

Without an explicit boundary condition at $r = 0$, the zero-coupon bond price function $\Phi(r, \tau)$ of the DSR model with time to maturity and market price of risk λ_r of state variable *X* (where $r = X^2$) is given by Longstaff (1989) as:[6]

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[4] It is a log utility general equilibrium model.

[5] The other two common characteristics include: (i) the interest rate returns to positive values if it approaches zero; and (ii) the instantaneous variance $\sigma_r^2 r$ is directly related to level of the interest rate. More detailed analyses and empirical evidence of stochastic interest rates following the DSR process, and the boundary behaviour of the process are in Longstaff (1989) and (1992), and Karlin and Taylor (1981. ch. 15).

[6] This is analogous to the unrestricted equilibrium discussed in section 3 in Longstaff (1992). Beaglehole and Tenney (1992) point out that Longstaff's (1989) bond pricing equation is not the solution for a reflecting boundary condition.

$$\Phi(r,\tau) = A(\tau)\exp\left\{C(\tau)\sqrt{r} + B(\tau)r\right\}, \tag{2}$$

where

$$A(\tau) = \sqrt{\frac{1-C_0}{1-C_0\exp\{\gamma\tau\}}}\exp\left(\alpha_1 + \alpha_2\tau + \frac{\alpha_3 + \alpha_4\exp\left\{\frac{1}{2}\gamma\tau\right\}}{1-C_0\exp\{\gamma\tau\}}\right), \tag{3}$$

$$B(\tau) = \frac{2\lambda_r - \gamma}{\sigma_r^2} + \frac{2\gamma}{\sigma_r^2\left[1-C_0\exp\{\gamma\tau\}\right]}, \tag{4}$$

$$C(\tau) = \frac{2\kappa_r(2\lambda_r+\gamma)\left(1-\exp\left\{\frac{1}{2}\gamma\tau\right\}\right)^2}{\gamma\sigma_r^2\left[1-C_0\exp\{\gamma\tau\}\right]}, \tag{5}$$

in which

$$\gamma = \sqrt{4\lambda_r^2 + 2\sigma_r^2} \quad , \quad C_0 = \frac{2\lambda_r+\gamma}{2\lambda_r-\gamma}$$

$$\alpha_1 = -\frac{\kappa_r^2}{\gamma^3\sigma_r^2}(4\lambda_r+\gamma)(2\lambda_r-\gamma) \quad , \quad \alpha_2 = \frac{2\lambda_r+\gamma}{4} - \frac{\kappa_r^2}{\gamma^2}$$

$$\alpha_3 = \frac{4\kappa_r^2}{\gamma^3\sigma_r^2}\left(2\lambda_r^2 - \sigma_r^2\right) \quad , \quad \alpha_4 = -\frac{8\kappa_r^2\lambda_r}{\gamma^3\sigma_r^2}(2\lambda_r+\gamma) \quad .$$

Another interesting property of the DSR model is that its associated zero-coupon bond price function $\Phi(r,\tau)$ is not a monotonically decreasing function of the interest rate *r*, and there exists a peak at a particular interest rate. Therefore, an increase in the interest rate does not always result in a decrease in the bond price.

Following the Black-Scholes-Merton's structural framework, a firm’s value *V* is assumed to follow a lognormal diffusion process and it’s liabilities *K* are governed by a mean-reverting lognormal diffusion. Their continuous stochastic movements are modelled by the following stochastic differential equations:

$$\frac{dV}{V} = \mu_V dt + \sigma_V dz_V \tag{6}$$

$$\frac{dK}{K} = \left[\mu_K + \kappa_K(\ln V - \ln K)\right]dt + \sigma_K dz_K \quad , \tag{7}$$

where $_{V}$ and $_{K}$ are the respective volatility values, $_{V}$ and $_{K}$ are the respective drift rates, the firm's liabilities *K* are mean-reverting at speed $_{K}$, and both dz_V and dz_K denote a standard Wiener process. When:

$$\ln K < \ln V + \frac{\mu_K - \frac{1}{2}\sigma_K^2}{\kappa_K} \quad , \tag{8}$$

the firm acts to increase $\ln K$, and vice-versa. This means that the firm tends to issue debt when its leverage ratio falls below a target level and reduces its liabilities when its leverage ratio is above target. Future changes in the liability structure of the firm generate uncertainty in the value of liabilities. There is no other explicit relationship assumed between the firm's value and the value of the liabilities.

Using the DSR model of the risk-free interest rate in the pricing framework means that the Wiener processes dz_r, dz_V and dz_K in Eqs. (1), (6) and (7) are correlated with:

$$dz_V dz_K = \rho_{VK} dt \tag{9}$$

$$dz_V dz_r = \rho_{Vr} dt \tag{10}$$

$$dz_K dz_r = \rho_{Kr} dt \, . \tag{11}$$

We define $L \equiv K/V$ to be the leverage ratio and apply Ito's lemma to derive the partial differential equation (PDE) governing a corporate discount bond $P(L,r,\tau)$ with the time-to-maturity of as follows:

$$\begin{aligned} \frac{\partial P(L,r,\tau)}{\partial \tau} &= \frac{1}{2}\sigma_L^2 L^2 \frac{\partial^2 P(L,r,\tau)}{\partial L^2} + \rho_{Lr}\sigma_L\sigma_r L\sqrt{r}\frac{\partial^2 P(L,r,\tau)}{\partial r \partial L} \\ &+ \frac{1}{2}\sigma_r^2 r \frac{\partial^2 P(L,r,\tau)}{\partial r^2} + \kappa_K\left[\ln\theta_L(\tau) - \ln L\right]L\frac{\partial P(L,r,\tau)}{\partial L} \, , \\ &+ \left(\frac{1}{4}\sigma_r^2 - \kappa_r\sqrt{r} - 2\lambda_r r\right)\frac{\partial P(L,r,\tau)}{\partial r} - rP(L,r,\tau) \end{aligned} \tag{12}$$

where

$$\sigma_L \equiv \sqrt{\sigma_V^2 - 2\rho_{VK}\sigma_V\sigma_K + \sigma_K^2} \, ,$$

$$\rho_{Lr} \equiv \frac{\rho_{Kr}\sigma_K - \rho_{Vr}\sigma_V}{\sigma_L}$$

$$\theta_L \equiv \exp\left[\left(\sigma_V^2 - \rho_{VK}\sigma_V\sigma_K - \delta\right)/\kappa_K\right]$$

$$\delta \equiv \left(\lambda_K\sigma_K + \mu_V - \mu_K\right) - \lambda_V\sigma_V .$$

Here θ_L is the target leverage ratio, and the terms $\lambda_K\sigma_K$ and $-\lambda_V\sigma_V$ are included to account for the risk premium attached to the firm's liabilities and value. Under the risk-neutral measure, is equal to zero. To obtain reasonable parameter values for the target leverage ratio, the dynamics of *V* and *K* are based on actual data such that is non-zero.

Under a scenario in which default can only occur at maturity and the recovery rate is *R* at default, the final payoff condition of the bond with face value 1 is:

$$P(L,r,0) = 1 \text{ if } L \le 1;$$

$$P(L,r,0) = R \text{ if } L > 1. \tag{13}$$

Using the derviation of the bond price solution in Appendix B, the integral in Eq. (B.6) can be straightforwardly evaluated to give the solution:

$$\begin{aligned} P(x,y,\tau) &= \Phi(x,\tau)\tilde{P}(x,y,\tau) \\ &= \frac{\Phi(x,\tau)}{\sqrt{2\pi\Delta(\tau)}}\int_{-\infty}^{0}\exp\left\{-\frac{\left[Y(x,y,\tau)-y'\right]^2}{2\Delta(\tau)}\right\}dy + \\ &\quad \frac{R\Phi(x,\tau)}{\sqrt{2\pi\Delta(\tau)}}\int_{0}^{\infty}\exp\left\{-\frac{\left[Y(x,y,\tau)-y'\right]^2}{2\Delta(\tau)}\right\}dy \quad , \\ &= \Phi(x,\tau)\left\{(1-R)N\left(-\frac{Y(x,y,\tau)}{\sqrt{\Delta(\tau)}}\right)+R\right\} \end{aligned} \tag{14}$$

where

$$x = \sqrt{2r} \quad , y = \ln\left(\frac{L}{L_0}\right) \quad , y_0 = \ln L_0 \quad , \tilde{\sigma}_r = \frac{\sigma_r}{\sqrt{2}} \quad , \tilde{\kappa}_r = \frac{\kappa_r}{\sqrt{2}} \quad , \kappa_L \equiv \kappa_K , \tag{15}$$

$$Y(x,y,\tau) = \exp\{-\kappa_L\tau\}y + \omega(\tau)\exp\{\zeta(\tau)\}x + \Omega_2(\tau) \ , \tag{16}$$

$$\Delta(\tau) = \int_0^\tau \sigma_y^2(\tau')d\tau' \quad , \tag{17}$$

$$\zeta(\tau) = \frac{1}{2}\gamma\tau + \ln\left|\frac{1-C_0}{1-C_0\exp(\gamma\tau)}\right| \quad , \tag{18}$$

$$\sigma_y^2(\tau) = \tilde{\sigma}_r^2 \omega^2(\tau) \exp\{2\zeta(\tau)\} + \sigma_L^2 \exp\{-2\kappa_L \tau\} + 2\rho_{Lr}\sigma_L \tilde{\sigma}_r \omega(\tau) \exp\{-\kappa_L \tau + \zeta(\tau)\}, \tag{19}$$

$$\omega(\tau) = \rho_{Lr}\sigma_L \tilde{\sigma}_r \int_0^\tau B(\tau') \exp\{-\kappa_L \tau' - \zeta(\tau')\} d\tau' , \tag{20}$$

$$\Omega_2(\tau) = \int_0^\tau \left[\frac{1}{\sqrt{2}} \rho_{Lr}\sigma_L \tilde{\sigma}_r C(\tau') + \kappa_L \ln \tilde{\theta}_L(\tau') \right] \exp\{-\kappa_L \tau'\} d\tau' + \int_0^\tau \left[\frac{1}{\sqrt{2}} \tilde{\sigma}_r^2 C(\tau') - \tilde{\kappa}_r \right] \omega(\tau') \exp\{\zeta(\tau')\} d\tau' , \tag{21}$$

$$\ln \tilde{\theta}_L(\tau) = \ln \theta_L(\tau) - y_0 - \frac{\sigma_L^2}{2\kappa_L} , \tag{22}$$

L_0 is a constant leverage ratio, and $N(.)$ denotes the cumulative distribution function of a standard normal distribution. The discount factor of the risky bond price $\tilde{P}(x, y, \tau)$ can be identified as a measure of default probability associated with a leverage ratio L ; that is, $P_{def}(L, r, \tau) = 1 - \tilde{P}(x, y, \tau)$. Furthermore, the credit spread $C_S(L, r, \tau)$ can be identified as:

$$C_S(L, r, \tau) = -\frac{1}{\tau} \ln\left(\frac{P(L, r, \tau)}{\Phi(r, \tau)} \right) = -\frac{1}{\tau} \ln\left(\tilde{P}(x, y, \tau)\right). \tag{23}$$

If we put $\kappa_L = 0$, the model is reduced to a pricing model in which the leverage ratio follows a lognormal process.

In order to allow default before maturity, a fixed or constant absorbing boundary (default barrier) is incorporated into the pricing model. When a firm's leverage ratio is above a predefined level, default occurs before maturity. This is consistent with an event of bankruptcy, associated with abnormally high levels of debt relative to the market value of the firm's assets. However, with a constant default barrier, no closed-form pricing solution is available. On the other hand, by the method of images, we can derive a closed-form pricing solution $\tilde{P}(x, y, \tau)$ which has a moving absorbing boundary specified by $y = y^*(x, \tau) \equiv \{\chi\Delta(\tau) - \Omega_2(\tau) - \omega(\tau)\exp[\varsigma(\tau)]x\}\exp[\kappa_L \tau]$ for some adjustable real parameter χ as follows:

$$\widetilde{P}(x,y,\tau)=\frac{1}{\sqrt{2\pi\Delta(\tau)}}\int_{-\infty}^{0}\left[\exp\left\{-\frac{[Y(x,y,\tau)-y']^{2}}{2\Delta(\tau)}\right\}-\exp(2\chi y')\exp\left\{-\frac{[Y(x,y,\tau)+y']^{2}}{2\Delta(\tau)}\right\}\right]f(y')dy'. \quad (24)$$

Since *f* (*y*) is equal to unity, the integral can be straightforwardly evaluated to give

$$\widetilde{P}(x,y,\tau)=N\left(-\frac{Y(x,y,\tau)}{\sqrt{\Delta(\tau)}}\right)-N\left(\frac{Y(x,y,\tau)}{\sqrt{\Delta(\tau)}}-2\chi\sqrt{\Delta(\tau)}\right)\times\exp\left[-2\chi Y(x,y,\tau)+2\chi^{2}\Delta(\tau)\right]. \quad (25)$$

With a recovery rate of *R*, the corporate bond price based on $\widetilde{P}(x,y,\tau)$ in Eq. (24) (no default before bond maturity) and Eq. (25) (default allowed before bond maturity) is:

$$P(L,r,\tau)=\Phi(x,\tau)\widetilde{P}(x,y,\tau)+R\left(1-\widetilde{P}(x,y,\tau)\right)\Phi(x,\tau) \quad (26)$$

As the movement of the absorbing boundary is adjustable by tuning the parameter χ, the default barrier can be adjusted such that the solution in Eq. (25) provides a good approximation to the exact result for a constant default barrier using the methodology developed in Lo et al. (2003) for solving barrier option values with time-dependent model parameters. In addition, such a dynamic default barrier is flexible and can incorporate different default scenarios as demonstrated in Hui et al. (2003). For example, default could be triggered even though a firm’s leverage ratio is below one (i.e., the firm’s asset value is above the total amount of debts issued by the firm) because of a liquidity problem (caused by repayment of short-term debt) faced by the firm. In this case, a default scenario which has a higher short-term default probability can be incorporated into the valuation model by adjusting the default barrier so that it is lower than one in the early period (say one to two years) of the time to maturity of the firm’s bond.

3. Characteristics and numerical results of corporate bond pricing model

As shown in Eqs. (25) and (26), the corporate bond pricing solution is a function of $Y(x,y,\tau)=\exp\{\xi(\tau)\}y+\omega(\tau)\exp\{\zeta(\tau)\}x+\Omega_2(\tau)$ where $x=\sqrt{2r}$. Since $\omega(\tau)$ is proportional to ρ_{Lr}, the credit spread depends explicitly upon the stochastic risk-free interest rate *r* if $\rho_{Lr}\neq 0$. Such dependence can be identified from the term $\rho_{Lr}\sigma_L\tilde{\sigma}_r B(\tau)x\frac{\partial\tilde{P}(x,y,\tau)}{\partial y}$ in Eq. (B.2), which is a nonlinear term arising from the DSR process of the interest rate. This is different to the pre-specified relationship between the interest rate and the leverage ratio in the pricing models of Longstaff and Schwartz (1995) and Colin-Dufresne and Goldstein (2001) in which Vasicek model is used for the interest rate dynamics in the pricing framework in Eq. (1).

To illustrate the effect of the level of the risk-free interest rate and the correlation coefficient between the leverage ratio and interest rates on credit spreads, Figure 1 shows the credit spreads of a 10-year bond issued by a medium leveraged (about BBB-rated) firm with a leverage ratio of 0.53 and $\sigma_L=0.28$. These credit spreads are calculated using the bond pricing model with $\kappa_L=0$ in Eqs. (14) and (23) for different ρ_{Lr}. The value of other model parameters σ_r^2, κ_r and λ_r of the DSR interest rate model are shown in Table 1. The DSR model parameters are estimated using Hansen's (1982) generalized method of moments technique which is also used by Longstaff (1989) to estimate a DSR term structure model.[7] The estimates in Table 1 are based on month-end zero-coupon yield to maturity data of 3-month, 12-month and 10-year US Treasury bills and notes during the January 1990 – December 2013 period.[8] The values of parameter σ_L used in Figures 1 and 2 for the leverage ratios across different credit ratings are based on the estimates by Feldhütter and Schaefer (2013) covering the period 2007Q3 – 2012Q2.

[7] Both the estimations in Table 1 and Longstaff (1989) have large standard errors, which are primarily due to the high correlation among the individual parameters.

[8] The data are from the US Department of the Treasury http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

As the credit spreads are obtained from a model with default at maturity, they are lower than those generated by models with default prior to maturity, i.e., with default barriers. On the other hand, assuming a zero recovery rate at default increases the level of credit spreads but does not change the dynamic relationship between credit spreads and the risk-free interest rate. Figure 1 shows that when the leverage ratio and interest rate are positively correlated ($\rho_{Lr} > 0$),credit spreads decrease with an increase in the interest rate. On the other hand, when $\rho_{Lr} < 0$, credit spreads increase with the interest rate. With $\rho_{Lr} = 0$, credit spreads remain the same with changes in the interest rate.

To understand the observations in Figure 1 which show the relationship between credit spreads and interest rate, we study a corporate bond pricing model which links the effective dynamics of a log-leverage ratio *y* and a risk-free interest rate *x* . From Eq. (B.2), the dynamics of the log-leverage ratio is expressed as:

$$dy = \kappa_L \left\{ \ln \tilde{\theta}_L(\tau) + \frac{1}{\kappa_L} \rho_{Lr} \sigma_L \tilde{\sigma}_r \left[\frac{C(\tau)}{\sqrt{2}} + B(\tau) x \right] - y \right\} d\tau + \sigma_L dz_y . \qquad (27)$$

Eq. (27) shows that the dynamics of the interest rate are linked to the dynamics of the leverage ratio nonlinearly into a mean-reverting term for the log-leverage ratio. The corresponding target leverage ratio is altered into an effective time-varying mean level $\left\{ \ln \tilde{\theta}_L(\tau) + \frac{1}{\kappa_L} \rho_{Lr} \sigma_L \tilde{\sigma}_r \left[\frac{C(\tau)}{\sqrt{2}} + B(\tau) x \right] \right\}$ which depends on the interest rate *x* explicitly. Therefore, the correlation coefficient between the leverage ratio and interest rate in the mean-reverting term has a material impact on credit spreads through the level of the interest rate.

As $B(\tau) < 0$, when $\rho_{Lr} > 0$, the time-varying mean level is a decreasing function of *x*, suggesting that the effective target leverage ratio will decrease with an increase in the level of the risk-free interest rate.[9] Thus, an increase in the level of the interest rate lowers the expected leverage ratio, implying lower credit spreads. This is consistent with the intuition that a central bank generally tends to increase interest rates in order to reduce inflationary pressures induced by increased investment and consumption in the country's economy. In such an economic environment, the profitability of

[9] The proof of $B(\tau) < 0$ is in Appendix A.

corporates should benefit from faster economic growth, implying that their asset values will increase and leverage ratios will decrease accordingly. Therefore, their reduced default risk is reflected in lower bond credit spreads. The empirical results in the next section using data from January 2008 to June 2013 support a negative relationship between credit spreads and interest rates. Given that the effect from the interest rate is an additional term in the drift of the leverage ratio, the occurrence of the effect does not depend on the choice of the dynamic process of the leverage ratio. This means that when the leverage ratio follows a lognormal process (i.e., no mean reversion), the interest rate dynamics continue to exert downward drift on the leverage ratio.

The analytical and numerical results of the corporate bond pricing solution suggest that a positive correlation between leverage ratios and risk-free interest rates is a necessary condition to give a negative relationship between credit spreads and interest rates. A positive correlation is supported by the observation that since early 2009, during which time the Federal Reserve has kept short-term interest rates near zero caused by an extremely weak economy in US, corporations have undertaken a deleveraging process. The debt of US non-financial corporations has fallen as a percent of GDP from 83% in March 2009 to 77% in March 2012.[10] During the same period, 10-year Treasury yields have declined from 2.82% to 2.17%, and further to 1.62% at the end of June 2012.

A positive correlation between leverage ratios and risk-free interest rates is also supported by the empirical relationship between equity values and interest rates. As a firm's leverage ratio is defined as total liabilities over its value, which is measured by the firm's market equity value plus total liabilities, liabilities appear in both the denominator and numerator of the leverage ratio. To some extent, changes in total liabilities cancel out in terms of their overall effect on the leverage ratio. Therefore, a positive correlation between leverage ratios and interest rates can be implied from a negative relationship between the equity values (which is in the denominator of the leverage ratio) and interest rates. Empirical studies, including Fama and Schwert (1977), Shiller and Beltratti (1992), Campbell and Ammer (1993), Domian et al. (1996) and Nissim and Penman (2003), have documented that equity returns are negatively related to changes in interest rates. An explanation of this relationship is that, according to economic theory, equity value is equal to the present value of expected risk-adjusted

[10] Debt includes all loans and credit market borrowing (e.g., bonds, commercial paper) and excludes asset-backed securities to avoid double counting of the underlying loan. Data are from US Federal Reserve Flow of Funds.

dividends discounted using the term structure of risk-free interest rates. Thus, holding constant expected risk-adjusted payoffs, an increase in interest rates reduces equity values. The empirical findings, which are consistent with the theory, suggest that the correlation between leverage ratios and risk-free interest rates is positive.

Conversely, the numerical results in Figure 1 shows that when $\rho_{Lr} < 0$, credit spreads will increase with an increase in interest rates as the expected leverage ratio rises. While the empirical results in the following section do not support this relationship between credit spreads and interest rates, this may happen to some corporates, which need to refinance their debts with higher interest rate payments when interest rates increase. Their asset values will decline due to higher refinancing costs. Accordingly, their leverage ratios and credit spreads are both expected to increase. Although these observations in general support $\rho_{Lr} > 0$, the condition of $\rho_{Lr} < 0$ occurs when corporates want to reduce their liabilities (i.e., their leverage ratios) because they consider refinancing too costly with rising interest rates.[11]

Regarding $\rho_{Lr} = 0$, the nonlinear effect arising from the DSR process of the interest rate disappears such that the effective time-varying mean level is simply $\ln \tilde{\theta}_L(\tau)$. The effect of the interest rate dynamics on credit spreads is basically reduced to that of the Vasicek and CIR models, i.e., no change in credit spreads with changes in the interest rates.[12] This means that the leverage ratio of a corporate and its credit spread are entirely independent of the level of interest rates. However, this finding is not supported by the empirical results reported in the next section.

Eq. (27) demonstrates that the sensitivity of credit spreads to the interest rate increases when the model parameters σ_r^2 and κ_r of the DSR interest rate process increase. An increase in the nonlinear effect of the DSR dynamics of the interest rate through the model parameters will change the effective time-varying mean level of the log-leverage ratio of a corporate further. This implies that the expected leverage ratio of the corporate and its corresponding credit spread become more dependent

[11] It is noted that Malitz (1994) reports that some firms tend to issue less debt in periods of high interest rates,

[12] The closed-form corporate bond price solution using the Vasicek model and the mean-reverting lognormal leverage ratio can be found in Hui et al. (2006).

on the level of interest rates.

To study the effect of the DSR process of interest rate dynamics on credit spreads for firms with different leverage ratios, Figure 2 shows the credit spreads of a low leveraged (AAA-rated) firm and a highly leveraged (BB-rated) firm with $\rho_{Lr} = -0.9$ and 0.9 using the same interest rate model parameters in Figure 1, which are compared with those of the BBB-rated bond. Given that the credit spreads of the AAA-rated firm are lower, changes in the credit spreads as interest rates change are smaller than those of the BBB-rated bond. However, their pattern of changes is similar, indicating that the nonlinearity of the interest rate has a material effect on bond pricess with a higher rating. Larger changes in the BB-rated bond's credit spreads with the interest rate relative to those of the AAA- and BBB-rated bonds reflect the notion that the default probability of a highly leveraged firm is more sensitive to the drift term in Eq. (27) associated with the log-leverage ratio. These results are supported by the empirical results in the next section: that the sensitivity of credit spreads to interest rates generally increases with lower credit ratings of the associated bonds. This is consistent with the intuition and observation that the default risk of highly leveraged corporates associated with lower credit ratings is more sensitive to the economic conditions reflected in the level of the risk-free interest rate. For example, both the actual default rates and credit spreads of speculative-grade bonds (i.e., BB+ or below) increased more substantially than those of investment-grade bonds (i.e., BBB- and above) in 2008 when the US Fed lowered the policy interest rate from the 4% level to 0-0.25%.[13]

4. Empirical relationship between credit spreads and risk-free interest rates

In this section, we conduct an empirical test on the relation between credit spreads and the risk-free interest rate using corporate bond yield data with different credit ratings over the period January 2008 to June 2013 when the risk-free interest rate was extremely low, as shown in Figure A1, following the global financial crisis. We regress changes in credit spreads on two factors following the method used in Longstaff and Schwartz (1995). We use monthly data for corporate bond yield averages across

[13] See Standard and Poor's (2009) "Default, Transition, and Recovery: 2008 Annual Global Corporate Default Study And Rating Transitions".

different credit ratings for the January 2008 to June 2013 period as well as the corresponding yields for 30-year Treasury bonds[14], resulting in a time series of 66 monthly observations. We use two sets of bond yield data: (1) US dollar domestic composite curves of S&P's AAA, AA, A, BBB, BB and B ratings from Bloomberg[15]; (2) Moody's Long-Term Corporate Bond Yield Averages of Aaa, Aa, A and Baa ratings[16]. There are 10 series of credit spreads.

For the interest rate, we use changes in the 30-year Treasury yield. For the return on the underlying assets, we use the returns on the S&P500 Index. Let ΔS denote the change in credit spreads, ΔY denote the change in the 30-year Treasury yield and *I* denote the return on the S&P500 Index. The regression equation is given by:

$$\Delta S = a + b\Delta Y + cI + \varepsilon , \tag{28}$$

where *a*, *b* and *c* are regression coefficients. Table 2 presents summary statistics for the credit spreads by credit ratings, 30-year Treasury yield and returns of the S&P500 Index.

Table 3 reports the regression results which suggest a negative relationship between credit spreads and risk-free interest rates. The estimated coefficients *b* are negative for each of the 10 credit spreads. With the exception of the AAA bonds, all of the estimates of *b* are statistically significant. The magnitude of the estimates of *b* shows that the relation between credit spreads and interest rates is both economically and statistically significant. For example, the regression results imply that a 100-basis point increase in the 30-year Treasury yield reduces AA-rated, BBB-rated and B-rated credit spreads by 33 basis points, 57 basis points, and 172 basis points respectively. This reflects a material relationship between credit spreads and risk-free interest rates during the low interest rate environment since the global financial crisis. The estimates using Moody's Long-Term Corporate Bond Yield Averages show similar results.

[14] The 30-year Treasury constant maturity series was discontinued on February 18, 2002, and reintroduced on February 9, 2006. From February 18, 2002, to February 9, 2006, the U.S. Treasury published a factor for adjusting the daily nominal 20-year constant maturity in order to estimate a 30-year nominal rate. Extrapolation of the series is done using the factor provided. The 10-year Treasury bonds are also used for our analysis. The results are similar to those from the 30-year Treasury bonds and available upon request.

[15] This is made up of individual curves of Industrials, Finance, Utilities and etc. However, the composition could change overtime when some of the individual curves discontinued. These series are put into groups according to their maturities, from 3 month to 30 year, but not all of them are available. Bloomberg selects bonds based on maturity, ratings, etc and builds the family of bonds from which a fair market curve is taken.

[16] Moody's Long-Term Corporate Bond Yield Averages are derived from pricing data on a regularly replenished population of corporate bonds in the US market, each with current outstanding values of over $100 million. The bonds have maturities as close as possible to 30 years; they are dropped from the list if their remaining life falls below 20 years, if they are susceptible to redemption, or if their ratings change. All yields are yield-to-maturity calculated on a semi-annual basis. Each observation is an unweighted average, with Average Corporate yields representing the unweighted average of the corresponding Average Industrial and Average Public Utility observations.

The coefficient *b* generally decreases with a higher credit rating of bonds. Bonds with a higher credit rating have a lower volatility of their leverage ratios according to the findings by Schaefer and Strebulaev (2008), and these results are consistent with Eq. (27) of a model where a decrease in volatility reduces the coefficient of *x* ($x = \sqrt{2r}$), thus, credit spreads less sensitive to the changes in the interest rate.

Table 3 also shows that all of the estimates of *c* are negative and statistically significant. The estimates of *c* decline monotonically in general with a higher credit rating of the bonds, with the exception of AAA-rated and Aa-rated credit spreads. The economic magnitude of these estimates is also important. For example, a 10 percent return (i.e., *I* = 0.10) reduces AA-rated, BBB-rated and B-rated credit spreads by 14 basis points, 20 basis points, and 70 basis points respectively. These results are consistent with the evidence of Jones et al. (1984), who find that equity returns are related to the prices of below-investment-grade bonds.

The results for *b* and *c* provide evidence that both the firm value and risk-free interest rate are factors determining credit spreads. To investigate their relative effects, we take a standard deviation of monthly changes in the 30-year Treasury yield over the sample period, which is 24 basis points. Similarly, the standard deviation of monthly returns for the S&P500 Index is 0.053. Multiplying these values by the parameter estimates *b* and *c* implies that a one-standard-deviation increase in the 30-year yield reduces the BBB-rated credit spread by 13.7 basis points, while a one-standard-deviation positive return for the S&P500 Index reduces the credit spread by 10.8 basis points. The corresponding measures for the Aa-rated credit spread are 6.4 basis points and 5.7 basis points, and the corresponding measures for the B-rated credit spread are 41.4 basis points and 37 basis points. Although the S&P500 Index is not a perfect measure of firm value, the above impact analysis demonstrates that the variation in credit spreads due to changes in the level of interest rates is comparable to the variation due to changes in the value of firms across different S&P's and Moody's credit ratings.

Finally, it is noted that the proposed model does not capture all of the variation in credit spreads. R^2s for the regressions range from 0.283 to 0.339 for the four credit spreads of Moody's Long-Term Corporate Bond Yield Averages. For the S&P's rating credit spreads, the regression R^2s are generally higher at 0.5 for low-rated (BB and B) bonds but less than 0.2 for A and higher grade bonds.

The results show a statistically and economically significant negative relationship between credit spreads and risk-free interest rates similar to the findings in previous studies and demonstrate such a relationship held during the low interest rate environment. The sensitivity of the credit spreads to interest rates generally decreases with a higher credit ratings of bonds.[17] This finding is consistent with the relationship between changes in credit spreads and the interest rate using different leverage ratios (i.e., credit ratings) shown in Figures 1 and 2. In general, as the volatility of the leverage ratios decreases with a higher credit ratings of the bonds, this result is consistent with Eq. (27) where a decrease in the volatility will make the coefficient of x smaller and thus the credit spread less sensitive to changes in the interest rate.

5. Conclusion

The evidence suggests that asset values or liabilities do not drift with interest rates, hence this paper develops a corporate bond pricing model following a structural approach in which the leverage ratio does not have any explicit relationship with the risk-free interest rate. By using a stochastic interest rate, which is governed by the double square-root (DSR) process in the pricing model, we derive a closed-form solution for corporate bond prices with default at maturity, and default prior to maturity with an adjustable moving default barrier respectively. As the interest rate under a DSR process is sticky downward, this feature is relevant to the low interest rate environment when the short rate is sticky at a level marginally above zero, as it has been since the global financial crisis in 2008.

The numerical results from the closed-form solutions show that credit spreads generated from our pricing model depend explicitly upon the level of the risk-free interest rate via a nonlinear effect arising from the DSR process if the correlation between the interest rate and leverage ratio is non-zero. Given a positive correlation between the interest rate and leverage ratio, the credit spreads decrease with an

[17] Similar empirical observations are also found in Longstaff and Schwartz (1995).

increase in the interest rate. This characteristic is consistent with the empirical findings using bond market data covering the period from 2008 to 2013. This demonstrates the importance of allowing for interest rate risk in addition to default risk in valuing corporate debt securities. In particular, the results provide supporting evidence for the empirical implications of incorporating a DSR model of the interest rate into a proposed corporate bond pricing model. Future research should focus on testing whether this corporate bond model is able to explain the actual level of corporate bond yields using detailed cross-sectional and time-series data for individual bonds and firms.

Appendix A

The Wiener process under the risk neutral probability (z_r^Q) for interest rate *r* can be written as

$$dz_r^Q = \xi_r dt + dz_r , \qquad \text{(A.1)}$$

where ξ_r is the market price of risk with respect to *r*. Eq. (1) under the historical probability measure becomes

$$dr = \left(\frac{\sigma_r^2}{4} - \kappa_r \sqrt{r} - \sigma_r \xi_r \sqrt{r} \right) dt + \sigma_r \sqrt{r} dz_r^Q \qquad \text{(A.2)}$$

under the risk neutral probability measure. Following Longstaff (1989) who used the market price of risk ${}_r$ with respect to the state variable *X* (where $X^2 = r$), the risk-adjusted Eq. (1) becomes

$$dr = \left(\frac{\sigma_r^2}{4} - \kappa_r \sqrt{r} - 2\lambda_r r \right) dt + \sigma_r \sqrt{r} dz_r^Q \qquad \text{(A.3)}$$

Eqs. (A.2) and (A.3) are equivalent if

$$\xi_r = 2 \frac{\lambda_r}{\sigma_r} \sqrt{r} . \qquad \text{(A.4)}$$

In terms of the new variable $x = \sqrt{2r}$, Eq. (A.3) can be written as

$$\begin{aligned} dx &= \left[\frac{\partial x}{\partial r} \left(\frac{\sigma_r^2}{4} - \kappa_r x - 2\lambda_r x^2 \right) + \frac{1}{2} \frac{\partial^2 x}{\partial r^2} \left(\sigma_r^2 x^2 \right) \right] dt + \frac{\partial x}{\partial r} \sigma_r x dz_x \\ &= \lambda_r \left(- \frac{\kappa_r}{\sqrt{2}\lambda_r} - x \right) dt + \frac{\sigma_r}{\sqrt{2}} dz_x \end{aligned} . \qquad \text{(A.5)}$$

Given $\kappa_r > 0$, for negative market price of risk (${}_r < 0$),

$$dx(t) = -|\lambda_r| \left(\frac{\kappa_r}{\sqrt{2}|\lambda_r|} - x \right) dt + \frac{\sigma_r}{\sqrt{2}} dz_x . \qquad \text{(A.6)}$$

If $x(0) < \kappa_r / \left(\sqrt{2} |\lambda_r| \right)$, $x(t)$ will move towards the origin. On the other hand, if $x(0) > \kappa_r / \left(\sqrt{2} |\lambda_r| \right)$, $x(t)$ will move towards $+\infty$. Therefore, if the current interest rate *r* is below the value $\left[\kappa_r / \left(\sqrt{2} |\lambda_r| \right) \right]^2$, then *r* will stick close to the origin.

As shown, the interest rate following the DSR process is sticky downward under a positive market price of risk and some conditions of a negative market price of risk. This feature is consistent with an extraordinarily low interest rate environment when the short rate is sticky at the level marginally above zero. To provide some evidence of this feature, Figure A1 shows that the three-month Treasury bill yield has been persistently near zero since late 2008 when the global financial crisis emerged. Table A1 reports the summary statistics for the yield. The first-order serial correlation of monthly changes is 0.484, reflecting that the yield is sticky and tends to be persistent at the low levels.

It is noted that $B(\tau) < 0$ in the bond price function $\Phi(r,\tau)$ expressed in Eq. (2) under the DSR model. Given that

$$e^{\gamma\tau} > 1 \qquad \text{(A.7)}$$

where

$$\gamma = \sqrt{4\lambda_r^2 + 2\sigma_r^2} > 0 \qquad , \qquad \tau > 0 ,$$

we can then write the following inequality:

$$\left(\frac{\gamma + 2\lambda_r}{\gamma - 2\lambda_r}\right) e^{\gamma\tau} > \frac{\gamma + 2\lambda_r}{\gamma - 2\lambda_r} \qquad \text{for } \gamma \pm 2\lambda_r > 0 . \qquad \text{(A.8)}$$

with

$$-C_0 = \frac{2\lambda_r + \gamma}{2\lambda_r - \gamma},$$

Eq. (A.8) can be expressed as

$$1 - C_0 e^{\gamma\tau} > 1 + \frac{\gamma + 2\lambda_r}{\gamma - 2\lambda_r} = \frac{2\gamma}{\gamma - 2\lambda_r} \qquad \text{(A.9)}$$

$$\Rightarrow \qquad \frac{2\gamma}{\sigma_r^2\left(1 - C_0 e^{\gamma\tau}\right)} < \frac{\gamma - 2\lambda_r}{\sigma_r^2} = -\frac{2\lambda_r - \gamma}{\sigma_r^2} \qquad \text{(A.10)}$$

$$\Rightarrow \qquad \frac{2\lambda_r - \gamma}{\sigma_r^2} + \frac{2\gamma}{\sigma_r^2\left(1 - C_0 e^{\gamma\tau}\right)} < 0 \qquad \text{(A.11)}$$

$$B(\tau) < 0$$

Appendix B

In terms of the new variables and parameters in Eq. (15), Eq. (12) becomes

$$\begin{aligned}\frac{\partial P(y,x,\tau)}{\partial \tau} &= \frac{1}{2}\sigma_L^2 \frac{\partial^2 P(y,x,\tau)}{\partial y^2} + \rho_{Lr}\sigma_L\tilde{\sigma}_r \frac{\partial^2 P(y,x,\tau)}{\partial x \partial y} \\ &+ \frac{1}{2}\tilde{\sigma}_r^2 \frac{\partial^2 P(y,x,\tau)}{\partial x^2} + \kappa_L\left[\ln\tilde{\theta}_L(\tau) - y\right]\frac{\partial P(y,x,\tau)}{\partial y}. \\ &- (\tilde{\kappa}_r + \lambda_r x)\frac{\partial P(y,x,\tau)}{\partial x} - \frac{1}{2}x^2 P(y,x,\tau)\end{aligned} \quad \text{(B.1)}$$

Without loss of generality, we assume $P(y,x,\tau)$ to be of the product form: $P(y,x,\tau) = \Phi(x,\tau)\tilde{P}(x,y,\tau)$, where the unknown function $\tilde{P}(x,y,\tau)$ denotes the discount factor of the risk-free bond price function $\Phi(x,\tau)$ due to the possibility of default. It is not difficult to show by direct substitution that $\tilde{P}(x,y,\tau)$ satisfies the PDE:

$$\begin{aligned}\frac{\partial \tilde{P}(x,y,\tau)}{\partial \tau} &= \frac{1}{2}\sigma_L^2 \frac{\partial^2 \tilde{P}(x,y,\tau)}{\partial y^2} + \rho_{Lr}\sigma_L\tilde{\sigma}_r \frac{\partial^2 \tilde{P}(x,y,\tau)}{\partial x \partial y} + \frac{1}{2}\tilde{\sigma}_r^2 \frac{\partial^2 \tilde{P}(x,y,\tau)}{\partial x^2} \\ &+ \left\{\frac{1}{\sqrt{2}}\rho_{Lr}\sigma_L\tilde{\sigma}_r C(\tau) + \rho_{Lr}\sigma_L\tilde{\sigma}_r B(\tau)x + \kappa_L\left[\ln\tilde{\theta}_L(\tau) - y\right]\right\}\frac{\partial \tilde{P}(x,y,\tau)}{\partial y}. \\ &+ \left\{\frac{1}{\sqrt{2}}\tilde{\sigma}_r^2 C(\tau) - \tilde{\kappa}_r - \left[\lambda_r - \tilde{\sigma}_r^2 B(\tau)\right]x\right\}\frac{\partial \tilde{P}(x,y,\tau)}{\partial x}\end{aligned} \quad \text{(B.2)}$$

In order to eliminate all the terms involving first derivatives with respect to x and y, we rewrite $\tilde{P}(x,y,\tau)$ as

$$\begin{aligned}\tilde{P}(x,y,\tau) &= \exp\left\{\zeta(\tau)x\frac{\partial}{\partial x}\right\}\exp\left\{\xi(\tau)y\frac{\partial}{\partial y}\right\}\exp\left\{\omega(\tau)x\frac{\partial}{\partial y}\right\}\times \\ &\exp\left\{\Omega_1(\tau)\frac{\partial}{\partial x}\right\}\exp\left\{\Omega_2(\tau)\frac{\partial}{\partial y}\right\}Q(x,y,\tau) \\ &= Q\left(\exp\{\zeta(\tau)\}x + \Omega_1(\tau), \exp\{\xi(\tau)\}y + \omega(\tau)\exp\{\zeta(\tau)\}x + \Omega_2(\tau), \tau\right)\end{aligned} \quad , \quad \text{(B.3)}$$

where

$$\Omega_1(\tau) = \int_0^\tau\left[\frac{1}{\sqrt{2}}\tilde{\sigma}_r^2 C(\tau') - \tilde{\kappa}_r\right]\exp\{\zeta(\tau')\}d\tau'$$

Then substituting Eq. (B.3) into Eq. (B.2), we can show that $Q(x,y,\tau)$ satisfies the two-dimensional diffusion equation:

$$\begin{aligned}\frac{\partial Q(x,y,\tau)}{\partial \tau} &= \frac{1}{2}\sigma_x^2(\tau)\frac{\partial^2 Q(x,y,\tau)}{\partial x^2} + \rho_{xy}(\tau)\sigma_x(\tau)\sigma_y(\tau)\frac{\partial^2 Q(x,y,\tau)}{\partial x \partial y} \\ &+ \frac{1}{2}\sigma_y^2(\tau)\frac{\partial^2 Q(x,y,\tau)}{\partial y^2}\end{aligned} \tag{B.4}$$

where

$$\sigma_x^2(\tau) = \tilde{\sigma}_r^2 \exp\{2\zeta(\tau)\} \ ,$$

$$\begin{aligned}\sigma_y^2(\tau) &= \tilde{\sigma}_r^2 \omega^2(\tau)\exp\{2\zeta(\tau)\} + \sigma_L^2 \exp\{-2\kappa_L \tau\} \\ &+ 2\rho_{Lr}\sigma_L \tilde{\sigma}_r \omega(\tau)\exp\{-\kappa_L \tau + \zeta(\tau)\}\end{aligned} ,$$

$$\begin{aligned}\rho_{xy}(\tau)\sigma_x(\tau)\sigma_y(\tau) &= \rho_{Lr}\sigma_L \tilde{\sigma}_r \exp\{-\kappa_L \tau + \zeta(\tau)\} + \\ &\tilde{\sigma}_r^2 \omega(\tau)\exp\{2\zeta(\tau)\} \ .\end{aligned}$$

It should be noted that since the final payoff condition is independent of the interest rate r (or equivalently x), i.e., $P(L,r,0)$ is a function of L only, it is obvious that $Q(x,y,0)$ does not depend upon x. Thus, by defining $f(y) = Q(x,y,0)$, we can readily obtain the solution of Eq. (B.4) as follows:

$$\begin{aligned}Q(x,y,\tau) &= \exp\left\{\frac{1}{2}\Delta(\tau)\frac{\partial^2}{\partial y^2}\right\} Q(x,y,0) \\ &= \frac{1}{\sqrt{2\pi\Delta(\tau)}}\int_{-\infty}^{\infty} \exp\left\{-\frac{(y-y')^2}{2\Delta(\tau)}\right\} f(y')dy'\end{aligned} \tag{B.5}$$

This solution satisfies the natural boundary condition. Hence, the corresponding solution $\tilde{P}(x,y,\tau)$ is given by

$$\tilde{P}(x,y,\tau) = \frac{1}{\sqrt{2\pi\Delta(\tau)}}\int_{-\infty}^{\infty} \exp\left\{-\frac{[Y(x,y,\tau)-y']^2}{2\Delta(\tau)}\right\} f(y')dy' , \tag{B.6}$$

The derviation of Eq. (B.4) is shown as follows. By using the definition of an exponential operator:

$$\exp(\hat{O}) = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{O}^n \tag{B.7}$$

and the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}\exp(-\hat{O})\hat{A}\exp(\hat{O}) &= \hat{A} + \frac{1}{1!}[\hat{A},\hat{O}] + \frac{1}{2!}\left[[\hat{A},\hat{O}],\hat{O}\right] + \\ &\frac{1}{3!}\left[\left[[\hat{A},\hat{O}],\hat{O}\right],\hat{O}\right] + \cdots\cdots\end{aligned} , \tag{B.8}$$

we have the following useful operator identities:

$$\exp(-\hat{O})\hat{A}\hat{B}\exp(\hat{O}) = \{\exp(-\hat{O})\hat{A}\exp(\hat{O})\}\{\exp(-\hat{O})\hat{B}\exp(\hat{O})\} \quad \text{(B.9)}$$

$$\exp\{-\eta(\tau)\hat{O}\}\frac{\partial}{\partial\tau}\exp\{\eta(\tau)\hat{O}\} = \frac{\partial}{\partial\tau} + \frac{d\eta(\tau)}{d\tau}\hat{O} \quad \text{(B.10)}$$

where the operator $\hat{O}$ is independent of τ

$$\exp\left\{-\zeta(\tau)x\frac{\partial}{\partial x}\right\}x\exp\left\{\zeta(\tau)x\frac{\partial}{\partial x}\right\} = \exp\{-\zeta(\tau)\}x \quad \text{(B.11)}$$

$$\exp\left\{-\zeta(\tau)x\frac{\partial}{\partial x}\right\}\frac{\partial}{\partial x}\exp\left\{\zeta(\tau)x\frac{\partial}{\partial x}\right\} = \exp\{\zeta(\tau)\}\frac{\partial}{\partial x} \quad \text{(B.12)}$$

$$\exp\left\{-\xi(\tau)y\frac{\partial}{\partial y}\right\}y\exp\left\{\xi(\tau)y\frac{\partial}{\partial y}\right\} = \exp\{-\xi(\tau)\}y \quad \text{(B.13)}$$

$$\exp\left\{-\xi(\tau)y\frac{\partial}{\partial y}\right\}\frac{\partial}{\partial y}\exp\left\{\xi(\tau)y\frac{\partial}{\partial y}\right\} = \exp\{\xi(\tau)\}\frac{\partial}{\partial y} \quad \text{(B.14)}$$

$$\exp\left\{-\omega(\tau)x\frac{\partial}{\partial y}\right\}\frac{\partial}{\partial x}\exp\left\{\omega(\tau)x\frac{\partial}{\partial y}\right\} = \frac{\partial}{\partial x} + \omega(\tau)\frac{\partial}{\partial y} \quad \text{(B.15)}$$

$$\exp\left\{-\omega(\tau)x\frac{\partial}{\partial y}\right\}y\exp\left\{\omega(\tau)x\frac{\partial}{\partial y}\right\} = y - \omega(\tau)x \quad \text{(B.16)}$$

$$\exp\left\{-\Omega_1(\tau)\frac{\partial}{\partial x}\right\}x\exp\left\{\Omega_1(\tau)\frac{\partial}{\partial x}\right\} = x - \Omega_1(\tau) \quad \text{(B.17)}$$

$$\exp\left\{-\Omega_2(\tau)\frac{\partial}{\partial y}\right\}y\exp\left\{\Omega_2(\tau)\frac{\partial}{\partial y}\right\} = y - \Omega_2(\tau) \quad \text{(B.18)}$$

We re-write Eq. (B2) as

$$\frac{\partial\tilde{P}(x,y,\tau)}{\partial\tau} = \hat{L}\tilde{P}(x,y,\tau) \quad \text{(B.19)}$$

where

$$\begin{aligned}\hat{L} &= \frac{1}{2}\sigma_L^2\frac{\partial^2}{\partial y^2} + \rho_{Lr}\sigma_L\tilde{\sigma}_r\frac{\partial^2}{\partial x\,\partial y} + \frac{1}{2}\tilde{\sigma}_r^2\frac{\partial^2}{\partial x^2} \\ &+ \left\{\frac{1}{\sqrt{2}}\rho_{Lr}\sigma_L\tilde{\sigma}_r C(\tau) + \rho_{Lr}\sigma_L\tilde{\sigma}_r B(\tau)x + \kappa_L\left[\ln\tilde{\theta}_L(\tau) - y\right]\right\}\frac{\partial}{\partial y} \\ &+ \left\{\frac{1}{\sqrt{2}}\tilde{\sigma}_r^2 C(\tau) - \tilde{\kappa}_r - [\lambda_r - \tilde{\sigma}_r^2 B(\tau)]x\right\}\frac{\partial}{\partial x} \quad .\end{aligned} \quad \text{(B.20)}$$

Then, substituting Eq. (B.3) into Eq. (B.19) yields

$$\begin{aligned}&\frac{\partial}{\partial\tau}\hat{U}(\tau)Q(x,y,\tau) = \hat{L}\hat{U}(\tau)Q(x,y,\tau) \\ \Rightarrow &\left\{\hat{U}(\tau)^{-1}\frac{\partial}{\partial\tau}\hat{U}(\tau)\right\}Q(x,y,\tau) = \left\{\hat{U}(\tau)^{-1}\hat{L}\hat{U}(\tau)\right\}Q(x,y,\tau)\end{aligned} \quad \text{(B.21)}$$

where

$$\begin{aligned}\hat{U}(\tau) = &\exp\left\{\zeta(\tau)x\frac{\partial}{\partial x}\right\}\exp\left\{\xi(\tau)y\frac{\partial}{\partial y}\right\}\exp\left\{\omega(\tau)x\frac{\partial}{\partial y}\right\}\times \\ &\exp\left\{\Omega_1(\tau)\frac{\partial}{\partial x}\right\}\exp\left\{\Omega_2(\tau)\frac{\partial}{\partial y}\right\}\end{aligned} \quad \text{(B.22)}$$

and $\hat{U}(\tau)^{-1}\hat{U}(\tau) = \hat{U}(\tau)\hat{U}(\tau)^{-1} = 1$. Finally, by applying the operator identities given in Eqs. (B.9)-(B.18) to Eq. (B.21), we can derive Eq. (B.4) in a straightforward manner.

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Figure 1: Credit spreads for a 10-year medium leveraged (BBB-rated) bond for different values of *r* and ρ_{Lr}. The parameter values used are L = 0.53, $\sigma_L = 0.28$, $\sigma^2 = 0.0152$, $\kappa_r = 0.0278$ and $\lambda_r = -0.0798$.

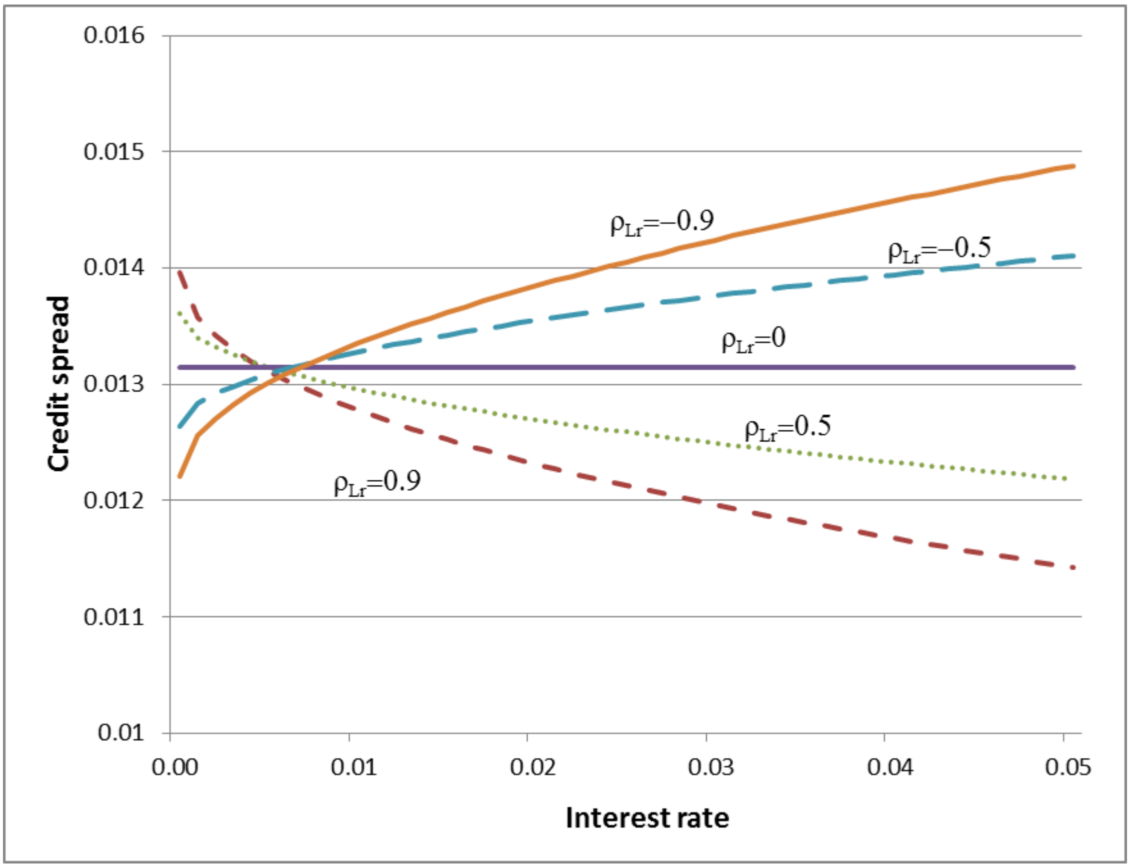

Figure 2: Credit spreads for 10-year low leveraged (AAA-rated: L = 0.29, $\sigma_L = 0.23$**), medium leveraged (BBB-rated:** L = 0.53, $\sigma_L = 0.28$**) and highly leveraged (BB-rated:** L = 0.71, $\sigma_L = 0.25$**) bonds for different values of** r **and** ρ_{Lr}. The parameter values used are $\sigma^2 = 0.0152$, $\kappa_r = 0.0278$ and $\lambda_r = -0.0798$.

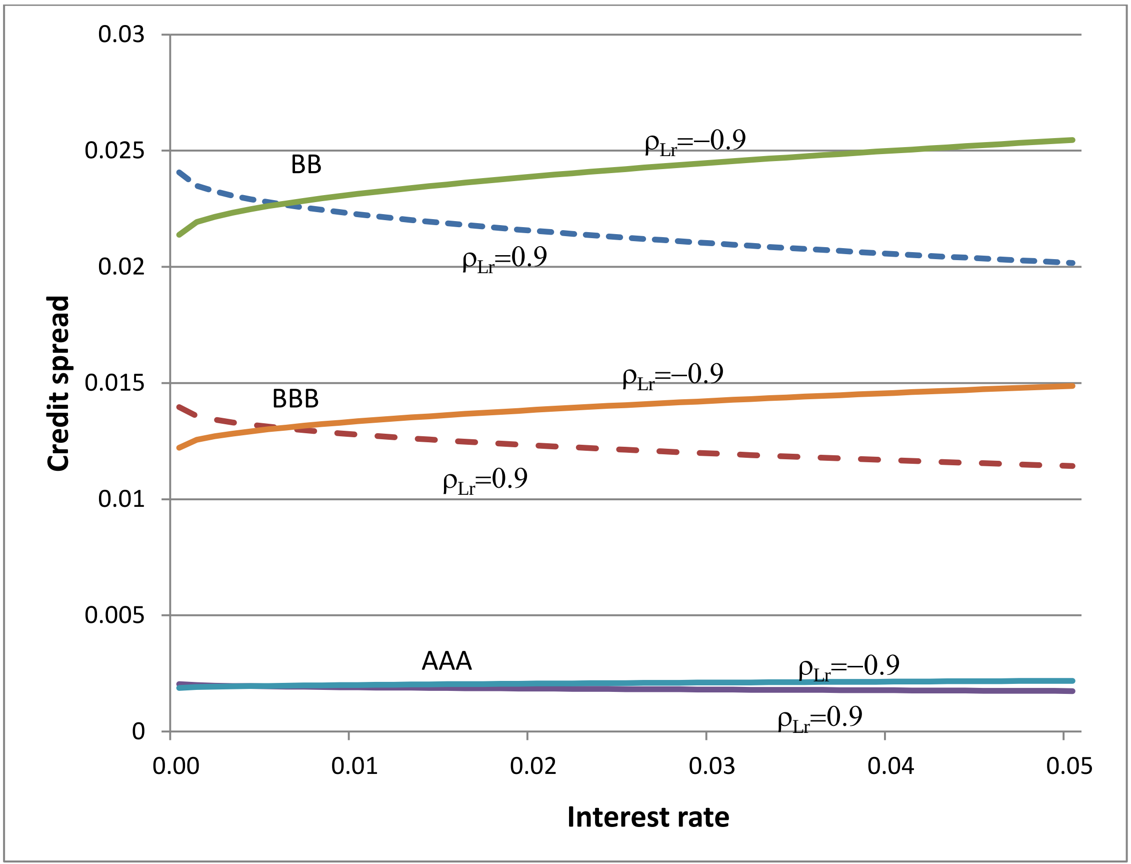

Figure A1: 3-month Treasury bill yield during 2008 to 2012.

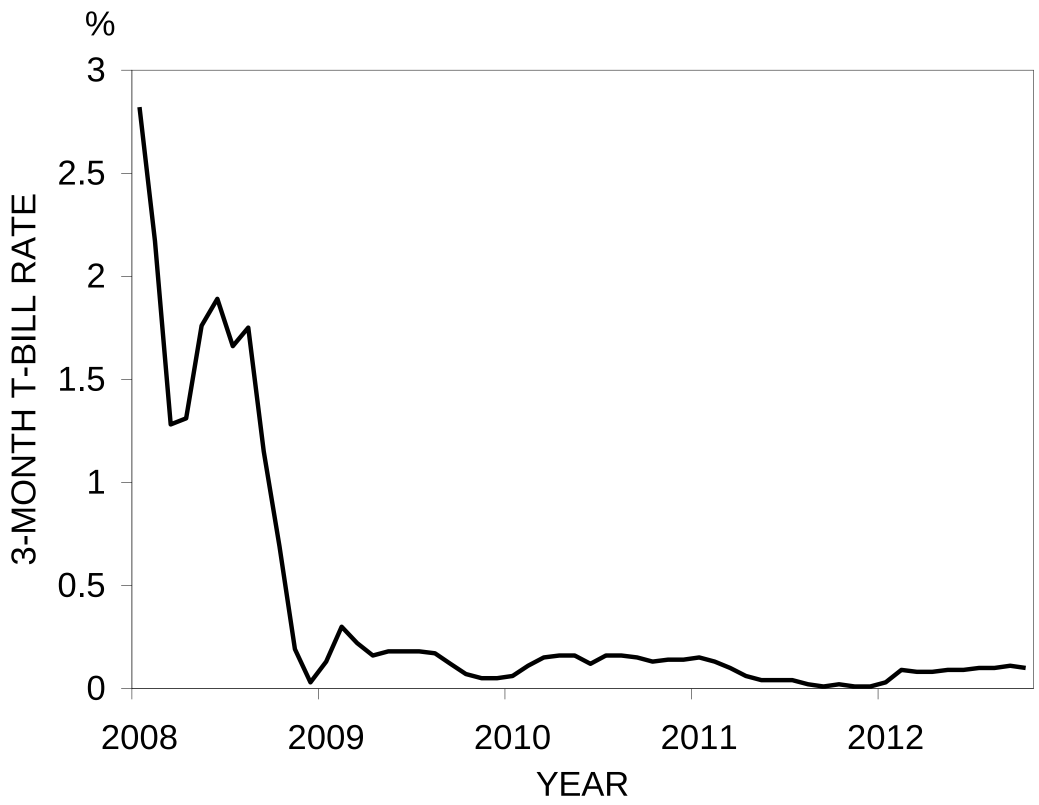

Table 1: Generalized method of moments estimates of the DSR model using month-end zero-coupon yield to maturity data of 3-month, 12-month and 10-year US Treasury bills and notes from January 1990 to December 2013.

Parameters	κ_r	σ_r^2	λ_r
Point estimate	0.0278	0.0152	-0.0798
Standard errors	0.0101	0.0015	0.1000

Note: The data are from the US Federal Reserve. The standard errors are computed by the Newey and West heteroskedasticity and autocorrelation-consistent estimate of the covariance matrix of the yields.

Table 2: Summary statistics for corporate bond credit spreads, US Treasury yield and returns on S&P500.

The credit spread is the difference between the corporate bond yield and the yield for a Treasury bond with the same maturity (30 year). Yields, spreads and returns on S&P500 are in percentage terms. The sample period is from January 2008 to June 2013.

	Mean	Standard Deviation	No. of Observations
Bloomberg Fair Value Credit Spread			
AAA	1.992	0.729	15
AA	1.563	0.758	51
A	1.916	0.796	66
BBB	2.500	1.121	66
BB	4.662	1.733	51
B	6.591	2.599	42
Moody's Credit Spread			
Aaa	0.938	0.390	66
Aa	1.194	0.565	66
A	1.542	0.692	66
Baa	2.325	0.998	66
30-year US Treasury Yield	3.818	0.673	66
Return on S&P 500 (Monthly)	0.278	5.308	66

Table 3: Results from regressing monthly changes in credit spreads on monthly changes in the 30-year US Treasury Yield and the returns on the S&P500 Index during January 2008 – June 2013.

$$\Delta S = a + b\Delta Y + cI + \varepsilon$$

ΔS: change in the credit spread
ΔY: change in the 30-year Treasury bond yield
I: return of S&P500

Bloomberg US dollar domestic composite curves

	a	b	c	t_a	t_b	t_c	R^2	N
AAA	-0.04615	-0.21644	-3.36383*	-0.34	-0.64	-1.88	0.232	15
AA	-0.01275	-0.33132*	-1.37816*	-0.30	-1.97	-1.85	0.159	51
A	-0.00774	-0.32729**	-1.62727**	-0.23	-2.28	-2.50	0.185	66
BBB	-0.01022	-0.57309***	-2.04587***	-0.29	-3.78	-2.97	0.313	66
BB	-0.00982	-1.27057***	-3.55823***	-0.18	-5.87	-3.71	0.551	51
B	-0.03800	-1.72352***	-6.98261***	-0.38	-4.34	-4.07	0.505	42

Moody's Long-Term Corporate Bond Yield Averages

	a	b	c	t_a	t_b	t_c	R^2	N
Aaa	-0.00304	-0.23350***	-0.83528***	-0.20	-3.69	-2.90	0.303	66
Aa	-0.00874	-0.26810***	-1.06622***	-0.53	-3.83	-3.35	0.339	66
A	-0.01086	-0.36448***	-1.06188**	-0.48	-3.78	-2.42	0.283	66
Baa	-0.00967	-0.56543***	-1.80078***	-0.29	-4.02	-2.81	0.321	66

Note: *, ** and *** indicate significant at 10%, 5% and 1% levels, respectively

Table A1: Summary statistics for 3-month Treasury bill yields and monthly changes in 3-month Treasury bill yields.

2008/1 – 2012/10	Yields	Changes in yields	Changes in yields p value
Mean	0.37328	-0.05121	
Std. dev.	0.63469	0.20197	
Minimum	0.01000	-0.89000	
Median	0.13000	0.00000	
Maximum	2.82000	0.45000	
ρ_1	0.815	0.484	(0.000)
ρ_2	0.652	-0.009	(0.001)
ρ_3	0.589	-0.213	(0.001)
N	58	58	

The data are from the US Federal Reserve.